Modulus functions

Modulus functions and their graphs.

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Modulus Functions

The modulus function or otherwise known as the absolute value of a real number *x* is defined by the following

$$x = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$

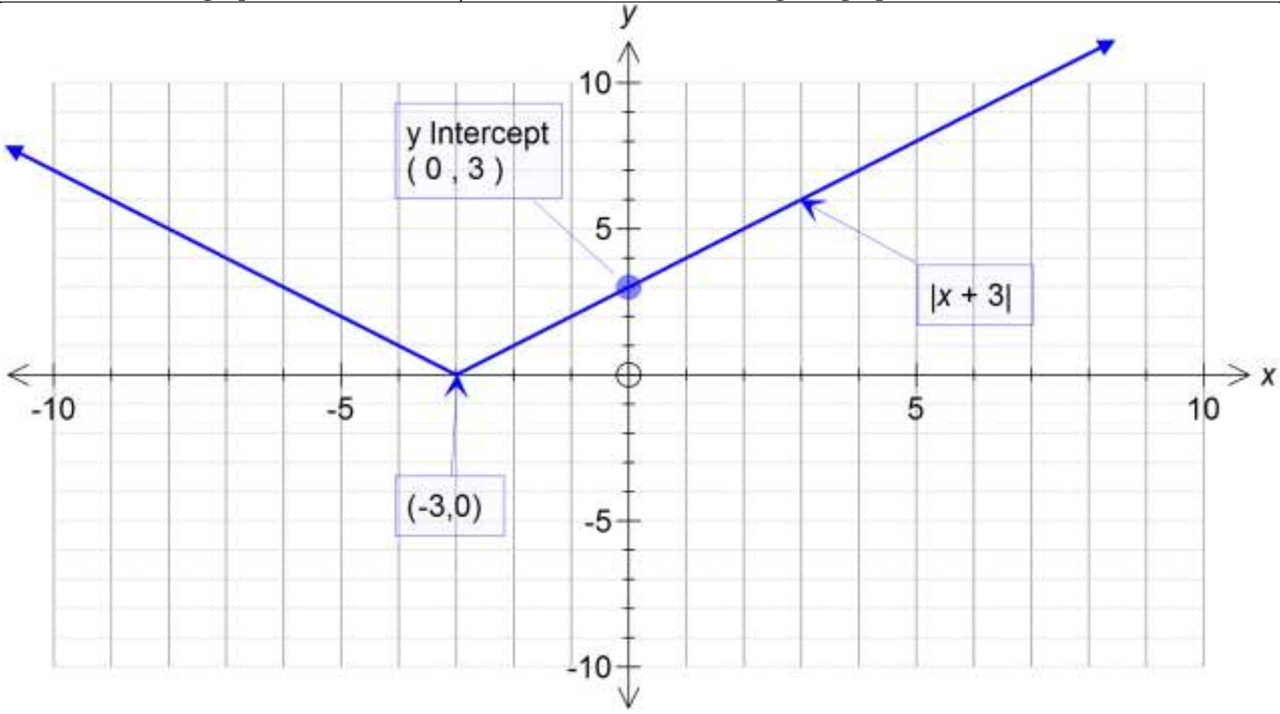
Properties of the Modulus Function

Property	Example
The absolute value of *x* is written as |x|. It is defined by the following:	$\|a\| = \begin{cases} (a) & if \; \text{a} \geq 0 \\ -(a) & if \; \text{a} < 0 \end{cases}$
$\|x\|$ can be thought of as the distance that *x* is from zero.	For example, the distance that 5 is from zero is 5, whereas the distance that -3 is from zero is 3. So we can then say $\|5\| = 5$ whereas $\|-3\| = 3$
If a and b are both non negative or both non positive then equality $\|a+b\| \leq \|a\| + \|b\|$	$\|3+5\| \leq \|3\| + \|5\|$
If $a \geq 0$ then $\|x\| \leq a$ is equivalent to $-a \leq x \leq a$	$\|x\| \leq 4$ Then we get the following expression $-a \leq x \leq a$
If $a \geq 0$ then $\|x-k\| \leq a$ is equivalent to $k-a \leq x \leq k+a$	$\|a-2\| \leq 4$ Then we get the following expression $2-a \leq x \leq 2+a$
$\|ab\| = \|a\| \bullet \|b\|$	
$\left\|\frac{a}{b}\right\| = \frac{\|a\|}{\|b\|}$	
Sometimes $\|x\|$ is referred to as magnitude of *x*, or the modulus of *x*, which can be thought to roughly mean the size of *x*	

Graphs of modulus

There are essentially a few ways to sketch modulus functions, namely we can use our graphic calculators (using the graph) or we can go from the definition method.
Let's us examine the definition method
Example 1: Sketch the graph of $y=|x+3|$

Steps	Method
First always start with the definition	Remember the definition $\lvert a \rvert \;=\; \begin{Bmatrix} (a) & if\ \ a\geq 0 \\ -(a) & if\ \ a<0 \end{Bmatrix}$
Now put in the required variables for the question at hand we get the following It pays to use the brackets so that we do not get confused	In this case we have $a=x+3$ So using the definition we have the following $\lvert x+3\rvert=\begin{Bmatrix} (x+3) & \text{if } x+3\geq 0 \\ -(x+3) & \text{if } x+3<0 \end{Bmatrix}$
Now we solve each expression separately.	$y=x+3 \text{ if } x+3\geq 0$ Now to solve this above equation we will have to remember how to deal with inequalities. Remember the inequality changes if we divide or multiply by a negative number! So we solve the inequality $x+3\geq 0$ $x\geq -3$ So this equation looks like this $y=x+3 \quad x\geq -3$
Now we solve the other inequality	Now we look at the other part of the expression $x+3<0$ $x<-3$ So this graph would be $y=-x-3 \quad$ for $x<-3$

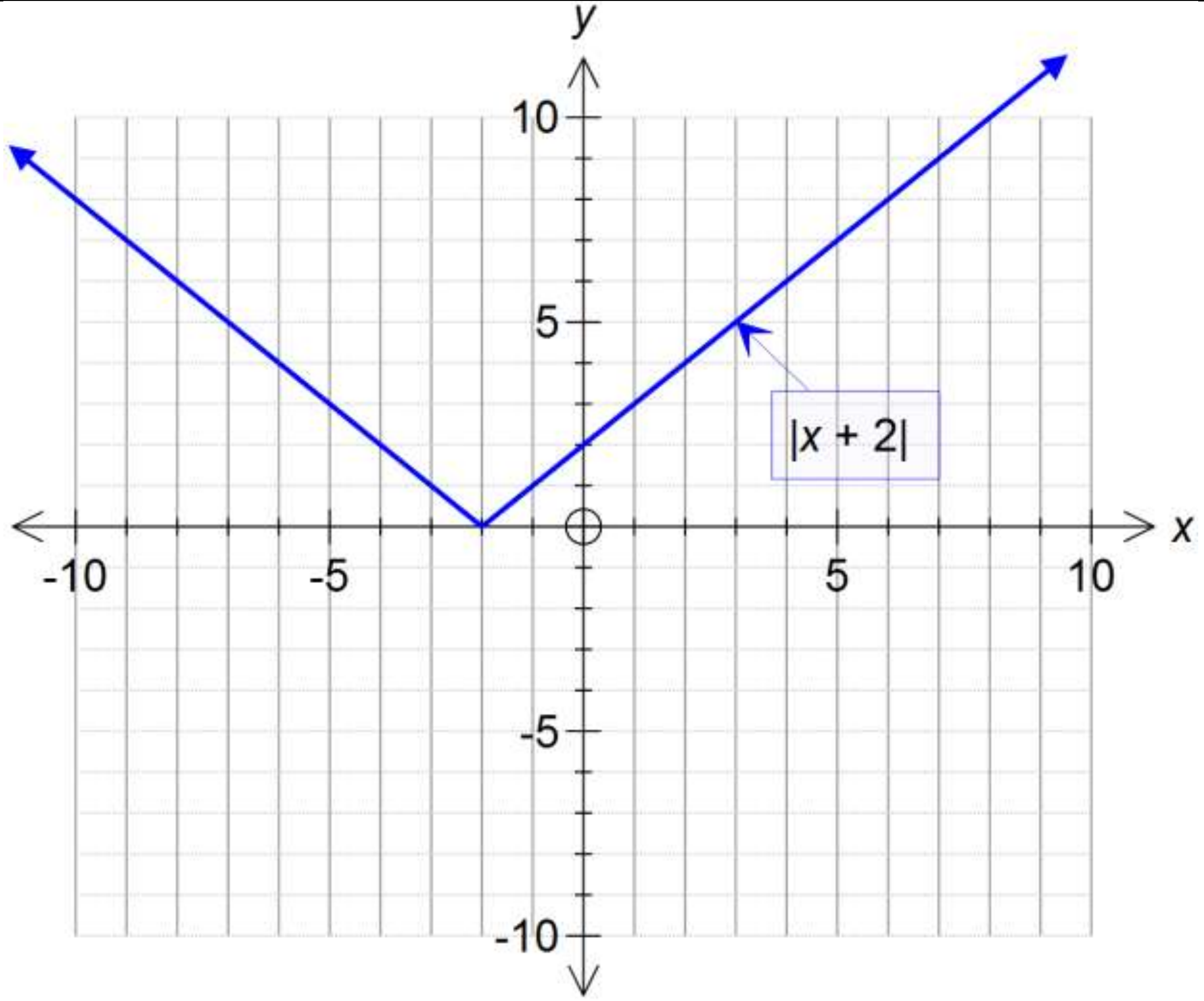
Now we sketch the graph	Be careful when sketching the graphs
	
Notice how the graph is positive for all values of x Now we could have just used our graphics calculators and we would had obtained the above graph quickly, however it is important to be able to do the maths.	

Let us look at a few more examples on using modulus functions.

Example 2: $y=|x+2|$

Start with the definition always	We use the definition of $\|a\| = \begin{cases} (a) & if\ \text{a} \geq 0 \\ -(a) & if\ \text{a} < 0 \end{cases}$ to see how to sketch this modulus function
Now replace the numbers with what we actually have	Here in the place of $a = x+2$ So we have the following $\|x-2\| = \begin{cases} (x+2) & if\ (x+2) \geq 0 \\ -(x+2) & if\ (x+2) < 0 \end{cases}$
Now separating the two expressions into two	$y = x+2$ if $(x+2) \geq 0$ which basically means $x \geq -2$
The other expression becomes	$y = -(x+2) \to -x-2$, which applies for the following $(x+2) < 0 \to x < -2$

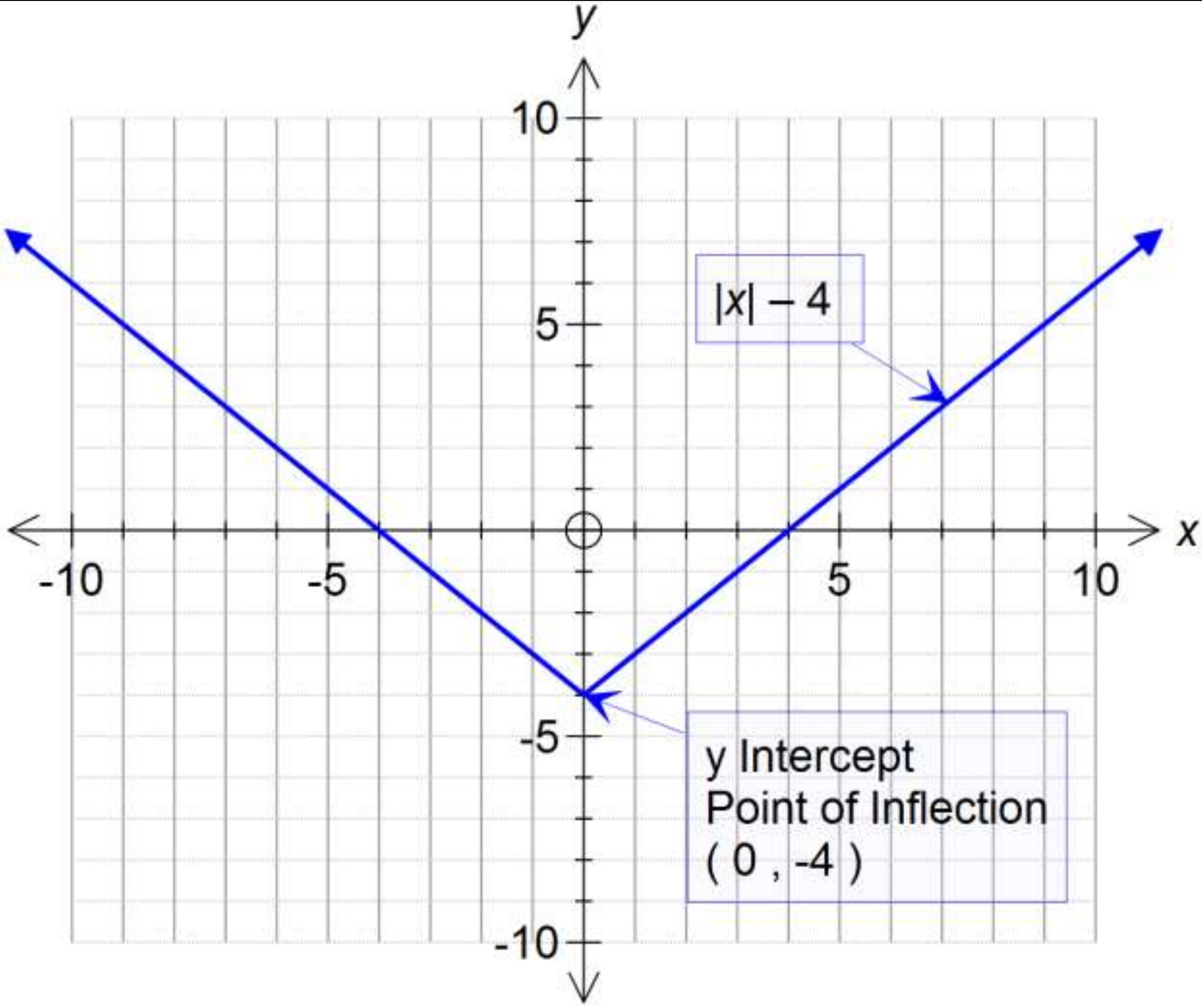
	Now we sketch the graph once again using both expressions



Notice how the next graph looks a little different; in the sense the modulus signs only cover the *x* values

Example 3: Sketch $y = |x| - 4$

Definition as always	Remember the definition $\lvert a \rvert = \left\{ \begin{array}{ll} a & if\ \text{a} \geq 0 \\ -a & if\ \text{a} < 0 \end{array} \right\}$
Now use the expressions for our question	$\lvert x \rvert = \left\{ \begin{array}{ll} x-4 & if\ \text{x} \geq 0 \\ -x-4 & if\ \text{-x} < 0 \end{array} \right\}$
Now do the first expression	Now the graph of $y = x - 4$ is for $x \geq 0$
Now the other expression	While the graph of $y = -x - 4$ if for $-x < 0$

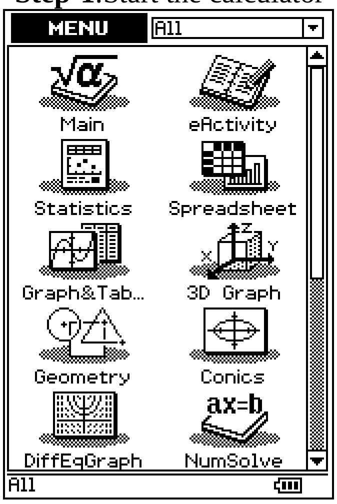


Notice how the graph has been moved downwards

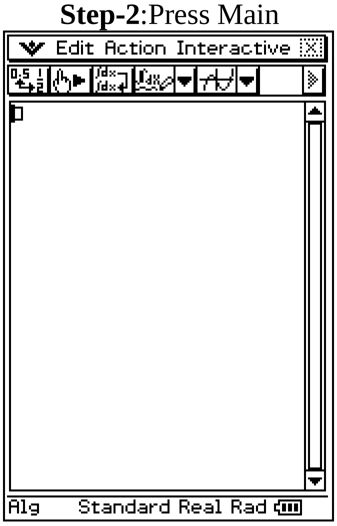
Using the graphics calculator to sketch the modulus functions

Many different ways let us look at two ways

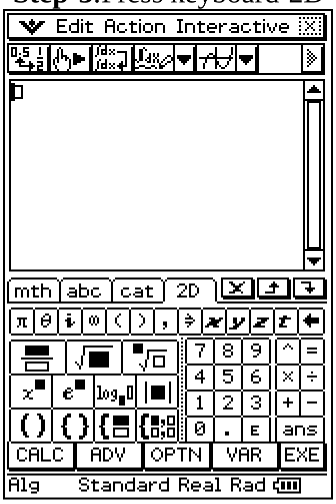
Step-1:Start the calculator

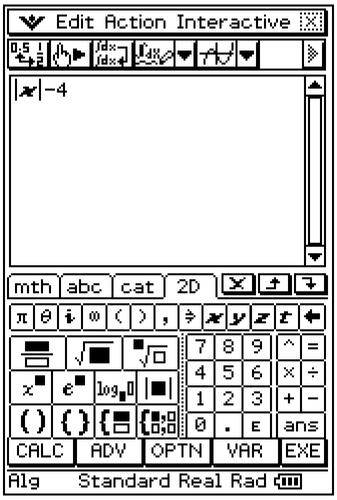
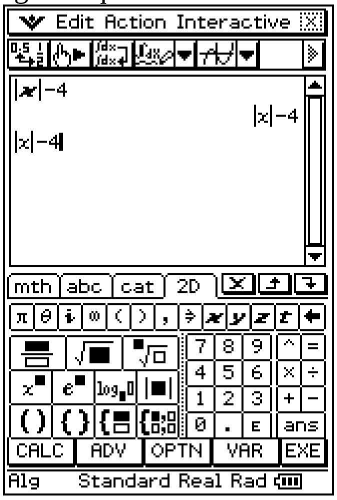
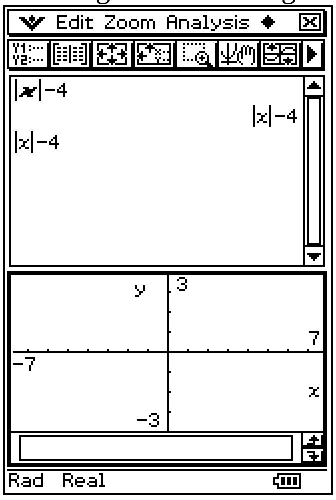
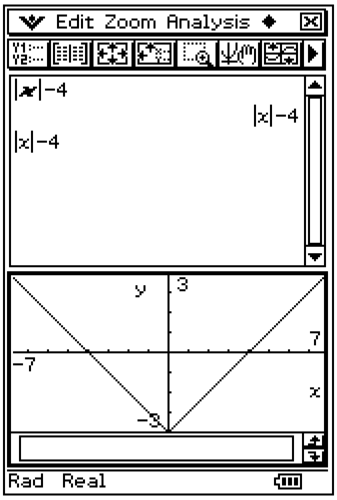
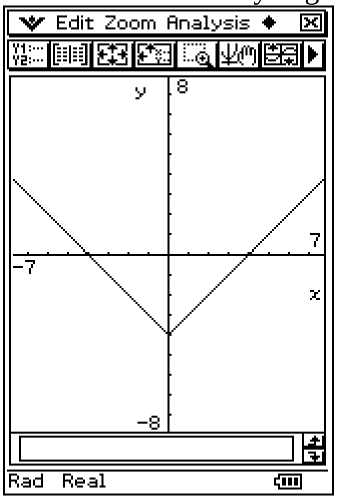
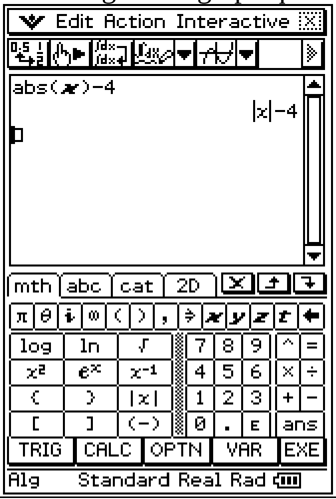


Step-2:Press Main



Step-3:Press keyboard-2D



Step-4:Press the absolute button and input the equation directly into the calculator 	**Step-5**:Press Enter the button and you will get the following. And drag the equation to the next line 	**Step-6**:Now press the graph button on the top of the screen and you will get the following 
Step-7:drag the equation to the screen on the bottom 	**Step-8**:Now click on the screen below and resize and you get 	**Step-9**:That is your graph You could had also done it by directly input the function from the screen using the short word **abs(x)-4** and moving to the graph quickly 

Skill Builder

Try the following questions to hone your skills

Sketch the graph of each of the following modulus function. Make sure you include the domain and range of the function,

a) Sketch the graph of $y=\lvert x+5\rvert$

b) Sketch the graph of $y=\lvert x\rvert+2$

c) Sketch the graph of $y=\lvert x-1\rvert$

d) Sketch the graph of $y=\lvert x-1\rvert+5$

e) Sketch the graph of $y=\lvert x+3\rvert-5$

f) Sketch the graph of $y=\lvert 2x-1\rvert+5$

g) Sketch the graph of $y=4-\lvert x\rvert$

h) Sketch the graph of $y=-\lvert x-2\rvert+1$

More difficult questions regarding modulus functions

How do we sketch the following graph? $y = \left| x^2 + x - 2 \right|$

Let us use the definition of modulus

$$|a| = \begin{cases} a & if \text{ a} \geq 0 \\ -a & if \text{ -a} < 0 \end{cases}$$

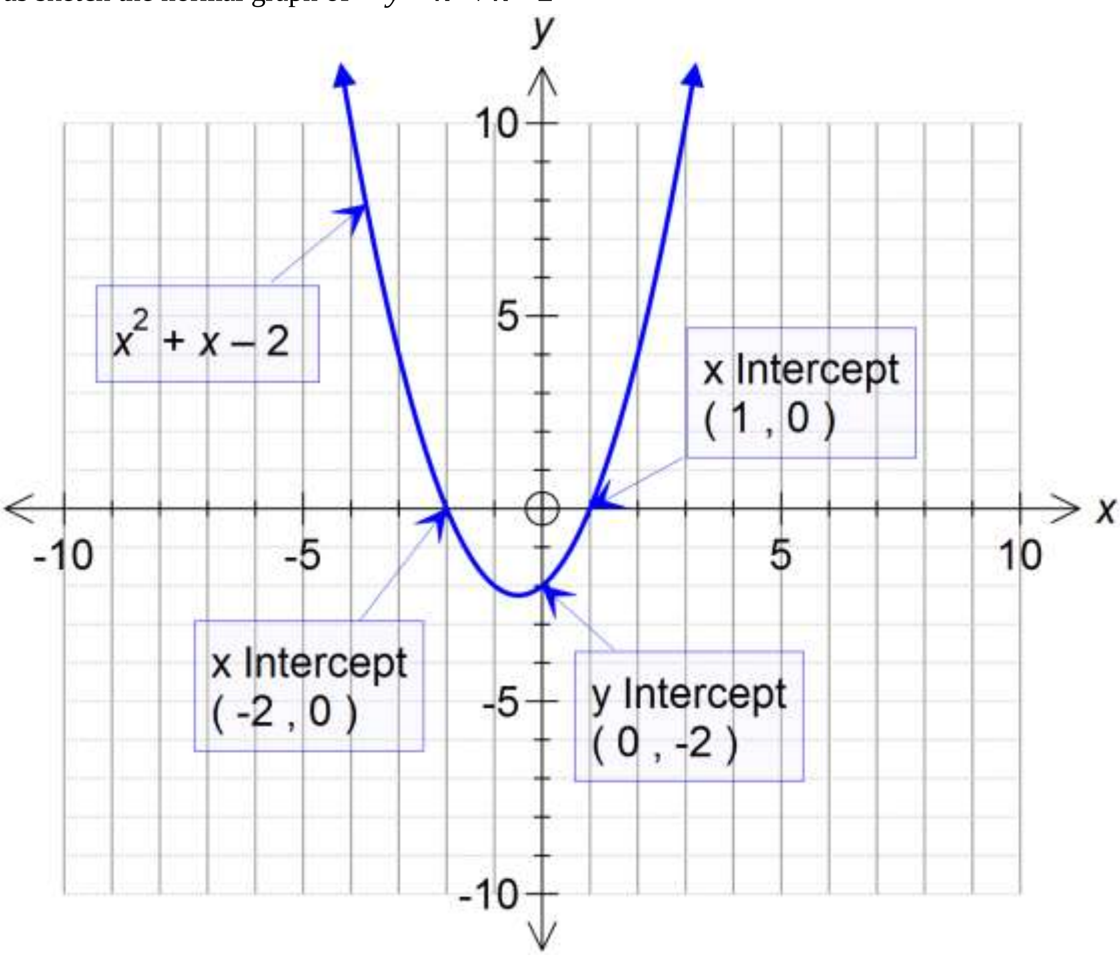
$$|x^2 + x - 2| = \begin{cases} \left(x^2 + x - 2\right) & if \ x^2 + x - 2 \geq 0 \\ -\left(x^2 + x - 2\right) & if \ x^2 + x - 2 < 0 \end{cases}$$

Now this is where it gets difficult

$x^2 + x - 2 \geq 0$

We can factorise the above quadratic equation, $x^2 + x - 2 \rightarrow (x+2)(x-1)$

Now lets us sketch the normal graph of $y = x^2 + x - 2$



From the above graph we can see that the x-intercepts are x= -2 and x=1
Now let us consider the two expressions to see when they are true

$$x^2 + x - 2 \geq 0$$
$$(x+2)(x-1) \geq 0$$

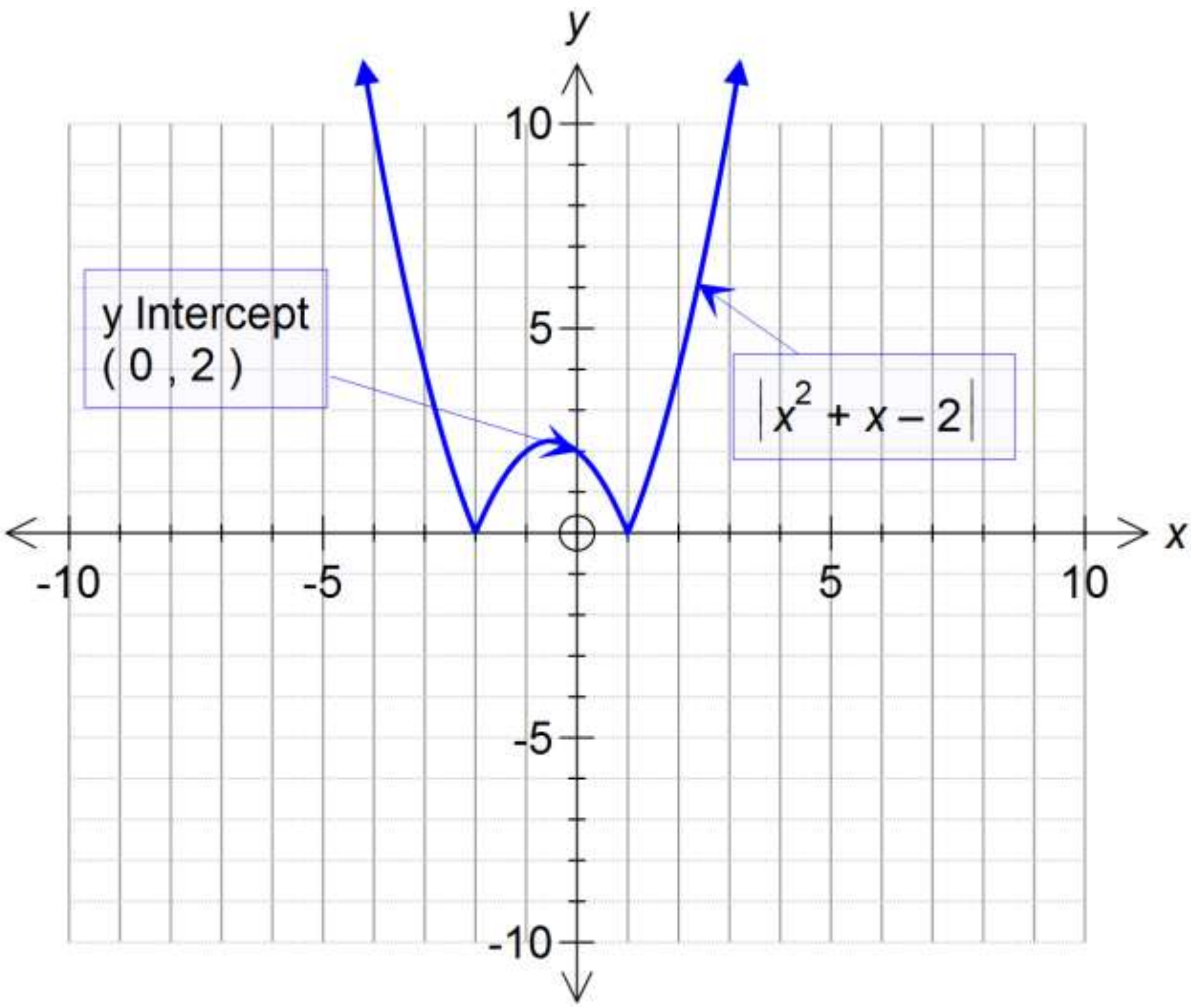
To get the above expression to be positive both brackets must be positive or both brackets must be negative, therefore $x \leq -2$ or $x \geq 1$

The second expression $-(x^2 + x - 2)$ if $x^2 + x - 2 < 0$

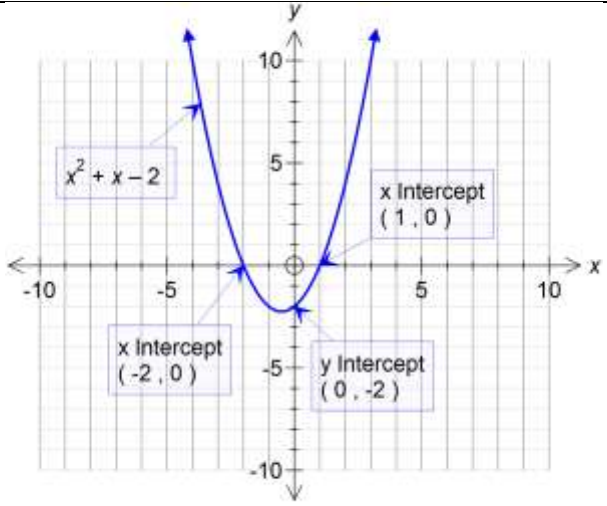
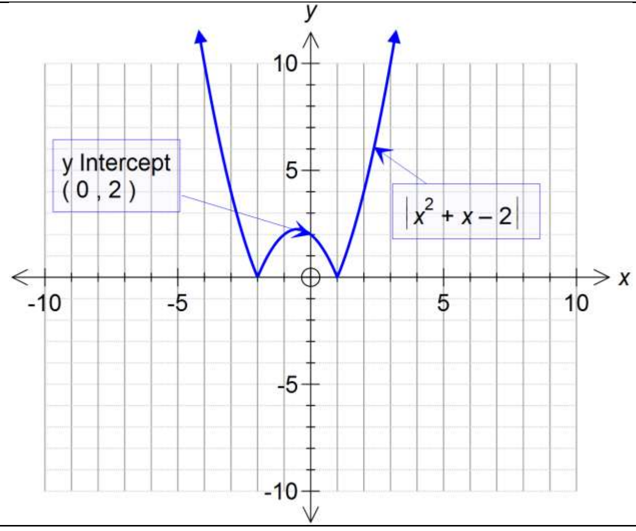
$$(x+2)(x-1) \geq 0$$

To get the above expression < 0 then x must be between -2 and 1, we write this like (-2, 1)

Then we can sketch the above graph taking into account the domain of the two expressions



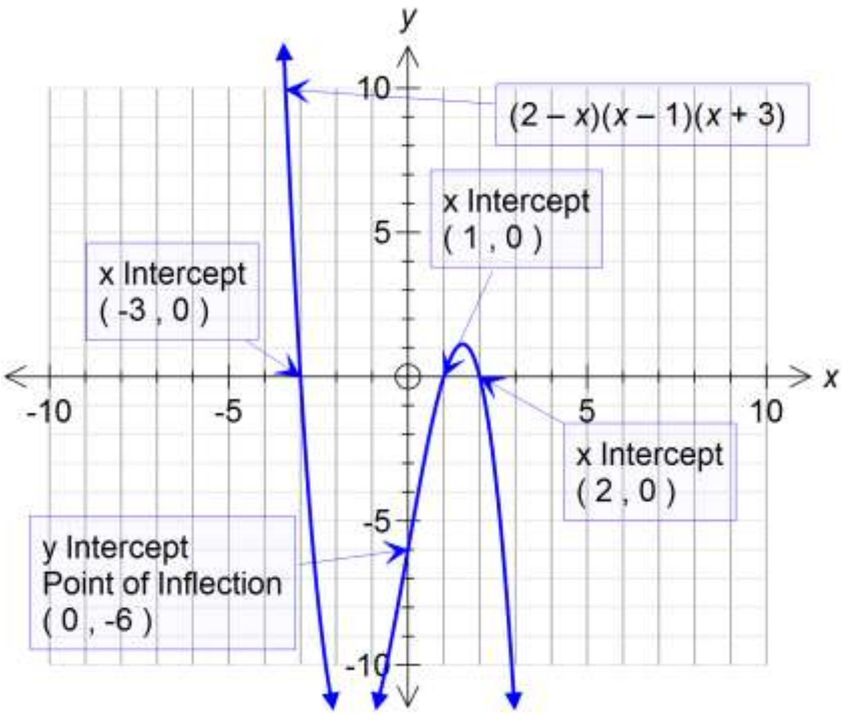
If you like at the two graph side by side notice what has actually happen.

	
The normal graph	Notice the how the bottom is reflected in the x-axis

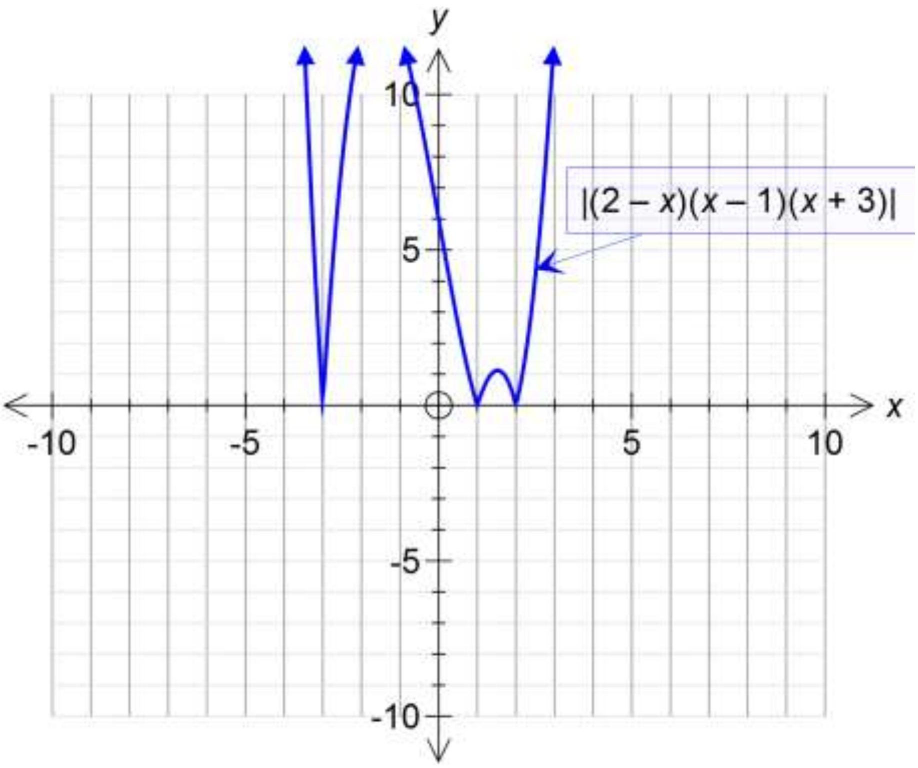
Method 2: Simply sketch the graph. $y = x^2 + x - 2$ The absolute value of a positive number is equal to that number, and the absolute value of a negative number is equal to the negative number and is therefore positive. So we simply sketch the normal graph and reflect in the x-axis the part of the graph that has a negative y value.

Example: Sketch the graph of $y = \lvert (2-x)(x-1)(x+3) \rvert$

Let's have done it the fast way



And if you reflect the



Skill builder

a) Sketch the graph of $y = \left|x^2 + 3x - 10\right|$

b) Sketch the graph of $y = \left|x^2 - 2x - 8\right|$

c) Sketch the graph of $y = \left|(x-1)(x+1)(x+3)\right|$

d) Sketch the graph of $y = \left|(x+1)(x+3)(x+4)\right|$

e) Sketch the graph of $y = \left|(x-1)^2(x-3)\right|$

f) Sketch the graph of $y = \left|(x+2)^2(x-2)\right|$

SOLVING MODULUS EQUATIONS

We normally use the absolute value in Physics when we are looking at the magnitude of something which basically means the value without worrying about it direction like in the case of velocity.
Simply put the absolute value means how far the number is from the zero on a number line.

$|6|$ its absolute value is 6

$|-6|$ Its absolute value is 6 also , as it is 6 units away from zero on the number line.

So in practice the 'absolute value' means to remove any negative sign and give its positive value.

Example

Problems	Answer
| 4|	4
|-12|	12
- | 4 |	-4 (tricky question, notice that the negative sign is outside the modulus sign)

Properties of the modulus

So when a number is positive or zero we leave it alone but if it is a negative number we change it to a positive

We can show that by using the following:

$$|x| = \begin{Bmatrix} (x) & \text{if } x \geq 0 \\ -(x) & \text{if } x < 0 \end{Bmatrix}$$

Which is the formal definition of what actually happens.

So |-17 | will give us the following -(-17) and we get 17

Another property which is extremely useful is the following:

$$\begin{aligned} &|z| = a \\ &z = \pm a \end{aligned}$$

Solve the following equation $|x+3| = 5$

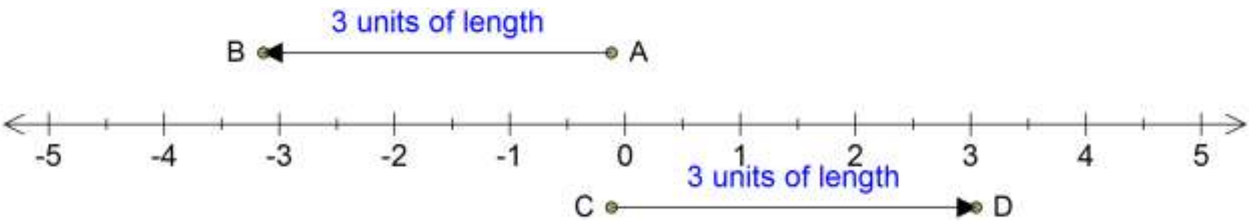
$$\begin{aligned} &|x+3| = 5 \\ &x+3 = \pm 5 \\ : \quad &x+3 = -5 \quad x+3 = 5 \\ &x = -8 \quad x = 2 \end{aligned}$$

Notice how we split the problems into two parts and then proceed with the individual solution

Let us return back to the definition of absolute value

$|x|$ is the distance of x from the zero.

So the modulus of $|3| = 3$ and so is $|-3| = 3$, so modulus is how far a number is from zero.



In summary then we have the following definitions:

Absolute values means	How far a number is from zero In Physics we tend to use the expression the magnitude of a number , it size
So if we are asked to find the absolute value of an number we just give a positive value	For example the absolute value of -30 is 30 Absolute value of -5 is 5 We tend to ignore the negative and just state the number
To show the absolute value	To show we are talking about the absolute value of a function we use the following |x|, we call them bars
Sometimes calculators we show the absolute values be using the expression abs(-1), which means find the absolute value of -1	Sometimes absolute value is also written as "abs()", so **abs(-1) = 1** is the same as **|-1| = 1**

Case 1- Questions involving < inequality or the $\leq$ inequality

The solution is always an interval and the pattern holds true.

The inequality to solve	What it means
$\|x\| < a$	$-a < x < a$
$\|x\| \leq a$	$-a \leq x \leq a$

Examples involving case 1 type of problems

Solve the following inequality $\|x\| < 5$	This means that the solution is giving by the above definition $-5 < x < 5$ and that is your answer However if you wanted the complete working out you will need to do the following: We split it to put parts from the definition of the modulus which is namely the following $\|x\| = \begin{Bmatrix} (x) & \text{if } x \geq 0 \\ -(x) & \text{if } x < 0 \end{Bmatrix}$ $x < 5,\ \text{provided } x \geq 0$ And the next part of the equation becomes $-x < 5$ $x > -5 \text{ provided } x < 0$ Now we have the two parts and we get the following inequality that provides us with the solution that is: $-5 < x < 5$ Now we could write this as an interval as $\left[-5, 5\right]$
Another example to illustrate the basic idea is solve for x the following	Let us use the quick way of solving this inequality It is of the form of case 1, so we get the following expression

$\lvert 2x-3\rvert \le 8$	$-8<2x-3<8$ $-8+3<2x-3+3<8+3$ $-5<2x<11$ $\frac{-5}{2}<x<\frac{11}{2}$ I could had separated the expressions and proceed with two equations but I decided in using the fast method to obtain the answer

Case 2- the inequality is $>$ or $\ge$

$\lvert x\rvert > a$	$x<-a$ or $x>a$ Remember that $\lvert x\rvert = \begin{Bmatrix} (x) & \text{if } x\ge 0 \\ -(x) & \text{if } x<0 \end{Bmatrix}$ So that means $-x>a$ $x<-a$
$\lvert x\rvert \ge a$	$x\le -a$ or $x\ge a$

So the solution is two inequalities not one. Do not try to combine them into one inequality as this would be a mistake.

Example

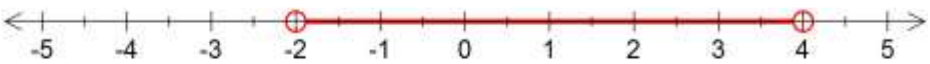
Solve $\lvert x\rvert > 2$	Solution $x<-2$ and $x>2$ Now we can write this as $(-\infty,-2)\cup(2,\infty)$ It would be a mistake to combine this and write is as $-2<x<2$? Why? You cannot have $x<-2$ and $x>2$ Hold true at the same time.
Solve $\lvert 2x-3\rvert > 5$	First step solve this equation $2x-3<-5$ $2x<-2$ $x<-1$ Then next step solve this equation $2x-3>5$ $2x>8$ $x>4$ And the answer is $x<-1\cup x>4$

What about if we are asked to go backwards?

Say for example- find the absolute value inequality that corresponds to the inequality of $-2 < x < 4$?

How would you proceed?

Let me show you a basic series of steps that you can use to get to the correct answer?

Step	Method
Sketch the number line and you will see the following	
Look at the end points	The end points are -2 and 4 and they are separated by 6 units apart
Now divide this by 2	Dividing by 2 we have 3
What is happening	I want to adjust the inequality so it is -3 and 3 instead of -2 and 4 For this to happen I will need to subtract 1 from both sides of the inequality
	$-2 < x < 4$ $-2-1 < x-1 < 4-1$ $-3 < x-1 < 3$
Does that look familiar	This is where you can look at case 1 and you can rewrite this as $\lvert x-1 \rvert < 3$, which is the answer

Skills builder

Show the following on a number line	$-3 < x < 4$ $-1 \le x < 5$ $(3 < x) \cup (x < -2)$
Simplify each of the following inequalities and draw a number diagram	$3x \le 9$ $5 - x < 7$ $5x - 1 \le 12 - 2x$
Find the solution sets of the following inequalities	$\lvert x-3 \rvert \le 11$ $\lvert 4x-3 \rvert \ge 12$ $\lvert 3x-1 \rvert > 5$

MODULUS FUNCTIONS			
1	What is modulus?	It can be thought of as distance from zero without worrying about positive or negative	$\lvert a \rvert = \begin{cases} (a) & if\ \text{a} \geq 0 \\ -(a) & if\ \text{a} < 0 \end{cases}$
2	What is the meaning of |3|?	It is called the modulus of 3 or the magnitude of 3 or the size of three which is 3 of course	$y = \lvert x-1 \rvert$ use the definition and then rewrite it as $\lvert x-1 \rvert = \begin{cases} (x-1) & x-1 \geq 0 \\ -(x-1) & x-1 < 0 \end{cases}$ Now split the problem into two , first do the top part $y=(x-1) \quad x-1 \geq 0$ and then do the second bottom part $y = -(x-1) \quad x-1<0$
3	How do you sketch graphs of modulus	Essentially you will need to split the graph up using the definition and then sketch each separately	
4	What do you need to be careful when working with modulus?	Watch out for the inequality signs carefully. If you multiply or divide by a negative number Also include the brackets!	Be careful to use a number line to help you with the domains and ranges also!