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Transformations

Transformations complete

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11/19/2012

Transformation Concepts

Transformations- General + Trigonometry-Matrices

Here is a summary that might be able to help you do these questions quickly-VERTICAL TRANSFORMATIONS
We start with the following function $f(x)$

Transformation	Meaning	What happens
Reflection in x-axis	Reflected in the x axis	$-f(x)$
Dilation of factor *a* from x axis	Vertical stretch by a factor *a*	$af(x)$
Translation of a units in the positive y direction	Vertical shift by a	$f(x)+a$
Translation of a units in the negative y direction	Vertical shift by a	$f(x)-a$

Now these I call HORIZONTAL TRANSFORMATIONS

Transformation	Meaning	What happens
Reflection in y-axis	Reflected in the y axis	$f(-x)$
Dilation of factor *a* from y axis	Horizontal shrink by a factor *a*	$f\left(\frac{x}{a}\right)$
Translation of a units in the positive x direction	Horizontal shift by a to the left	$f(x-a)$
Translation of a units in the negative x direction	Horizontal shift by a to the right	$f(x+a)$

Remember if you do not feel confident then I strongly suggest that you use the long algebraic method to work the finished transformation equation rather than mess up with the above short cuts.

Dilation of factor *k* from y axis	$(x,y)\to(kx,y)$
Dilation of factor *k* from x-axis	$(x,y)\to(x,ky)$
Translation of *k* units in the positive x-direction	$(x,y)\to(x+k,y)$
Translation of *k* units in the negative x direction	$(x,y)\to(x-k,y)$
Translation of *k* units in the positive y-direction	$(x,y)\to(x,y+k)$
Translation of *k* units in the negative y-direction	$(x,y)\to(x,y-k)$
Reflection in the x-axis	$(x,y)\to(x,-y)$
Reflection in the y-axis	$(x,y)\to(-x,y)$
Reflection in the line $y=x$	$(x,y)\to(y,x)$

Transformation- what is it?	Simply put a transformation simply changes an equation by applying either a dilation or a translation or a reflection. As a result the equation or graph will look different.
How many types of transformations do we have?	We have quite a few but essentially they break down into the following: Dilations Translations Reflections And combinations of the above
Is there a easy way to transform equations?	Yes but it involves going through the methodical algebraic methods which takes time. Most people learn to do it quickly by remembering the general ideas and applying them in one or two steps. In this section I will discuss both methods

Does it matter which transformation you apply first?	In theory no, but it is easier sometimes particularly with trigonometric functions to build up the transformation starting from inside the function and working outwards. Examples will show how this is done.
Let's us look at the transformations in detail to see what actually happens. We will move to use the full algebraic method which is the method you can use when you want to do it step by step and get the answer	
DILATIONS	
	We have 2 types of dilations; **1.** Dilation of factor k from the x-axis **2.** Dilation of factor k from the y-axis We will examine to see what each dilation does

Dilation of factor k from y axis

We will use the following equation for most of our transformations

Assume the original equation is

$y = \sin x$

Lets us look at the dilation of factor 2 from the y-axis

What this actually means and this is the hard thing to remember, that is why you will write this down on your bound book and with practice you will remember is that

A dilation of factor 2 from the y-axis has the following rule

$(x, y) \to (2x, y)$

Its effect would be to stretch out parallel to the x axis the graph

Now pay attention as the next few steps I will show you the full method

(x, y)	(x', y')
(x, y)	$(2x, y)$
Now the image is the $(2x, y)$	(x', y')
	We will equate the two expression above

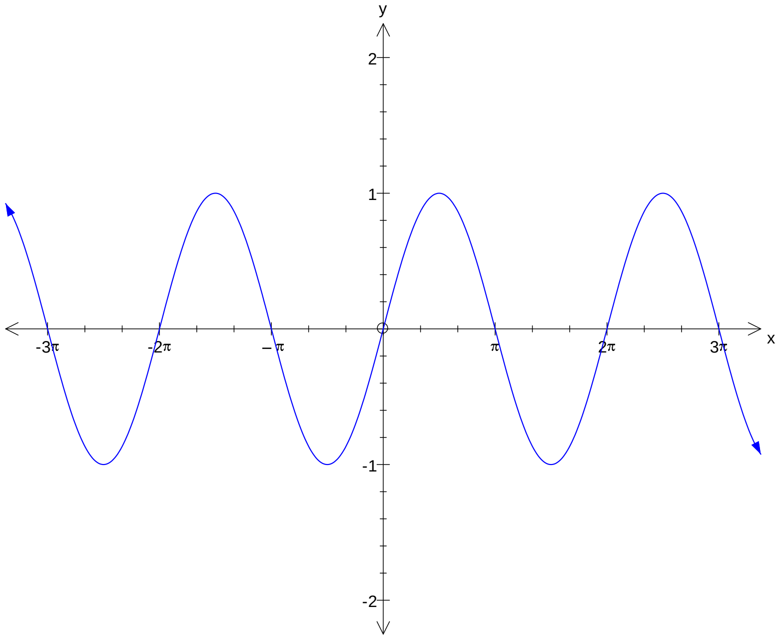
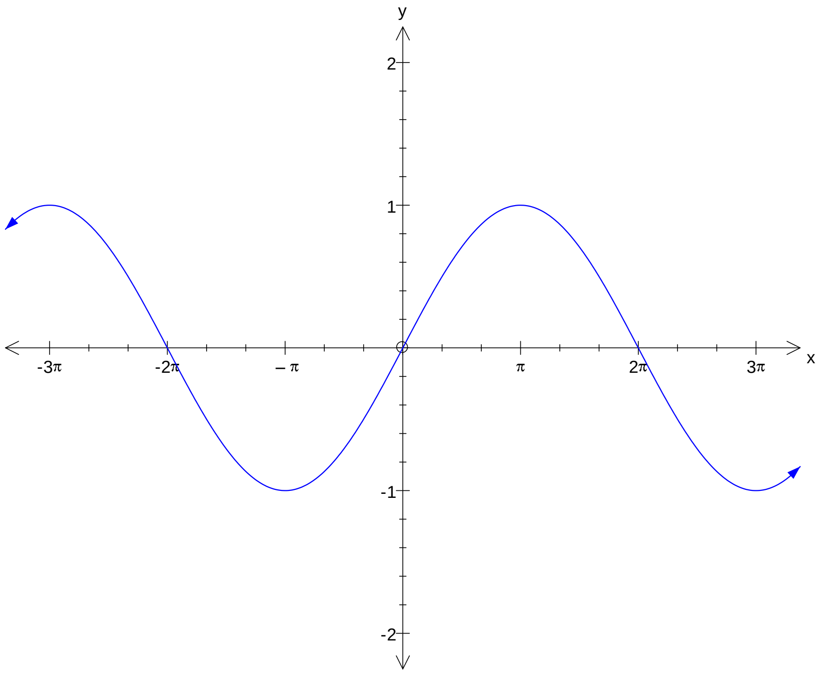
How would you work it out algebraically?

What steps we need to do	Steps taken
We equate the dashes with the transformation	Here $x' = 2x$ and $y' = y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = \frac{x'}{2}$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \sin\frac{x'}{2}$
Now we drop the dashes and we have	$y = \sin\frac{x}{2}$

This is the same as $y = \sin(\frac{1}{2}x)$

This is the full method

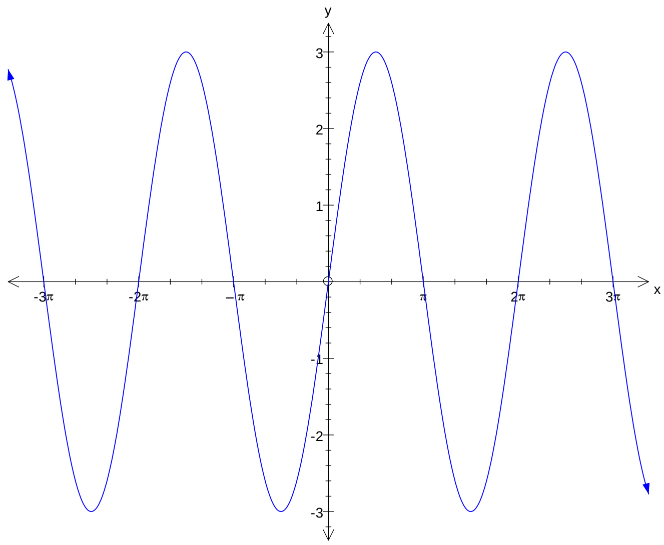
What does this transformation do to the original equation?	

<table>
<tr><td>Original equation</td><td>
</td></tr>
<tr><td>Now the transformation</td><td>
</td></tr>
<tr><td>So what has happened?</td><td>The graphs period has changed 2π to 4π
Why? The period is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$</td></tr>
<tr><td>Now let us examine what happens to a dilation of factor 3 from the x-axis</td><td>Next let us examine a dilation of factor 3 from the x-axis.
This has the following rule $(x, y) \to (x, 3y)$
<table>
<tr><td>(x, y)</td><td>(x', y')</td></tr>
<tr><td>(x, y)</td><td>$(x, 3y)$</td></tr>
<tr><td>Now the image is the $(x, 3y)$</td><td>(x', y')</td></tr>
<tr><td></td><td>We will equate the two expression above</td></tr>
</table>
This transformation has the effect of stretching the graph out parallel to the y-axis

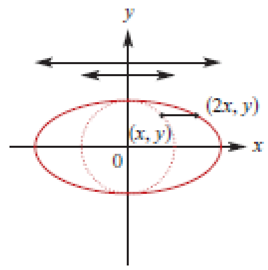
Now we use the long method</td></tr>
</table>

<table>
<tr>
<td></td>
<td>

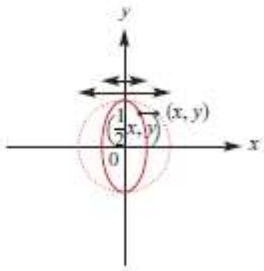
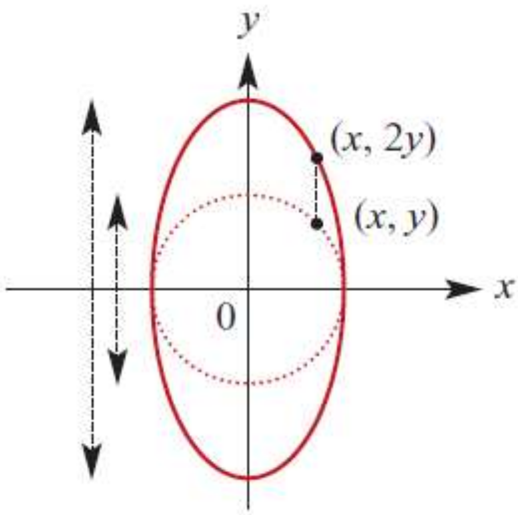
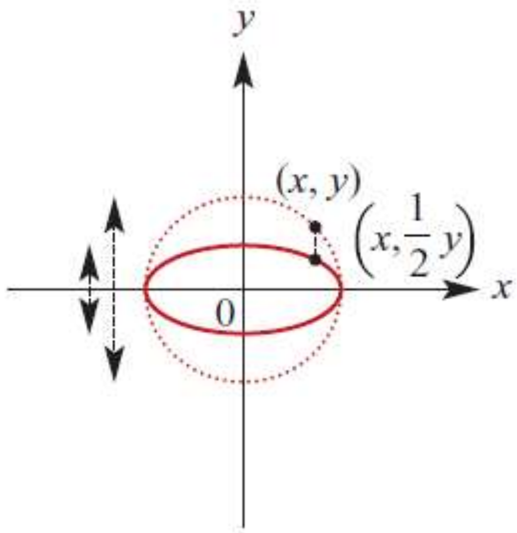
What steps we need to do	Steps taken
We equate the dashes with the transformation	Here $x' = x$ and $y' = 3y$
Rearrange x and y to be subject so that we can substitute into the original equation	$y = \frac{y'}{3}$ and $x = x'$
Once we put those expressions into the original equation we get the following	$\frac{y'}{3} = \sin x$
Now we drop the dashes and we have	$y = 3\sin x$

So the new transformed equation becomes
$y = 3\sin x$
</td>
</tr>
<tr>
<td>The new transformed equation looks like this</td>
<td>


Notice that the amplitude is 3 times the original equation
</td>
</tr>
<tr>
<td>So what do we notice about dilations ?</td>
<td>
Dilations of factor k from the y-axis

Examples
A dilation of factor 2 from the y-axis


The graph is stretched to twice the width
A dilation of factor $\frac{1}{2}$ from the y axis
</td>
</tr>
</table>

	 The graph is shrunk to half the width Dilations of factor k from the x-axis- for example a dilation of factor 2 from the x-axis  The graph is stretched to twice the height Dilation of factor $\frac{1}{2}$ from the x axis results in the following  Here the graph is shrunk to halve the height
Summary of Dilations	Dilation of factor k from the y-axis If $k > 1$ then graph is stretched in the x axis If $k < 1$ then graph is shrunk in the x axis Dilation of factor k from the x-axis

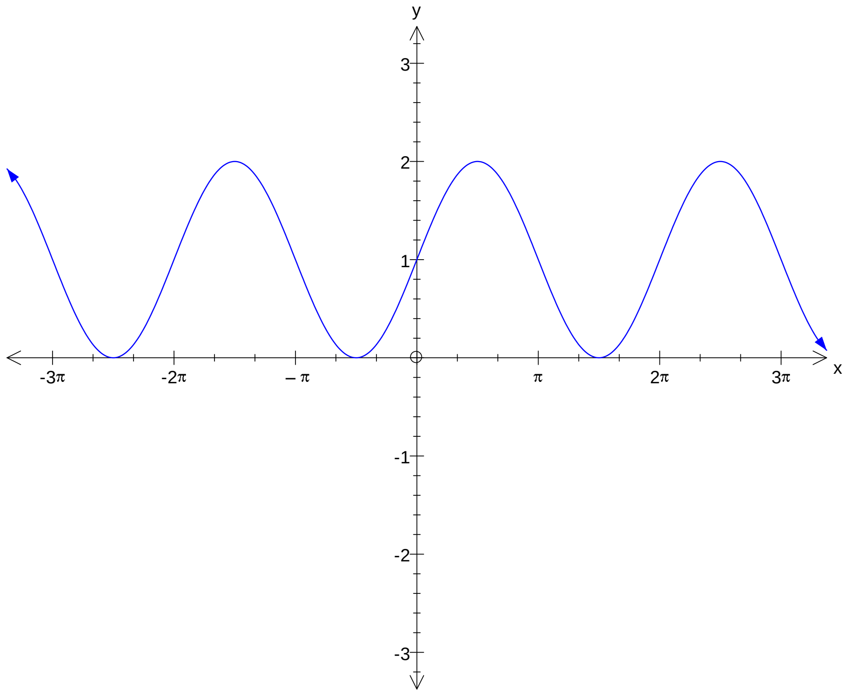
<table>
<tr><td></td><td>If $k>1$ then graph is stretched in the y axis
If $k<1$ then graph is shrunk in the y axis</td></tr>
<tr><td>Translations</td><td></td></tr>
<tr><td></td><td>Once again we have two types of translations:
Translation in the positive or negative direction of y axis
Translation in the positive or negative direction of x-axis</td></tr>
<tr><td></td><td></td></tr>
<tr><td colspan="2">Translation in the direction of the y-axis</td></tr>
<tr><td>We start with the original equation
Assume the original equation is
$y=\sin x$</td><td>Start with the original equation and translate of 1 unit in the positive direction of y means which should give us the following equation- $y=\sin x+1$

Now it is easy to see what will happen here
$(x,y)\to(x,y+1)$
So comparing to the
$(x,y)\to(x',y')$
$(x,y)\to(x,y+1)$
<table>
<tr><td>(x,y)</td><td>(x',y')</td></tr>
<tr><td>(x,y)</td><td>$(x,y+1)$</td></tr>
<tr><td>Now the image is the $(x,y+1)$</td><td>(x',y')</td></tr>
<tr><td></td><td>We will equate the two expression above</td></tr>
</table>

We see the following relationship
<table>
<tr><td>What steps we need to do</td><td>Steps taken</td></tr>
<tr><td>We equate the dashes with the transformation</td><td>Here $x'=x$ and $y'=y+1$</td></tr>
<tr><td>Rearrange x and y to be subject so that we can substitute into the original equation</td><td>$y=y'-1$
$x=x'$</td></tr>
<tr><td>Once we put those expressions into the original equation we get the following</td><td>$y'-1=\sin x'$</td></tr>
<tr><td>Now we drop the dashes and we have</td><td>$y-1=\sin x$
$y=\sin x+1$</td></tr>
</table>

So what happens then?
The original equation is moved upwards 1 unit in positive direction
Here is the equation of the new graph</td></tr>
</table>

Notice the equation now



Translation in the negative y axis of 1 unit

This would be the same as above except this time the translation is negative
For example say a translation of 1 unit in the negative direction of y would mean

$$(x, y) \to (x, y-1)$$

And we would follow the above steps to obtain the new transformed equation!

Translation in the direction of the x-axis

Translation of 1 unit in the positive x direction

The transformation is now

$$(x, y) \to (x+1, y)$$

So comparing to the

$$(x, y) \to (x', y')$$

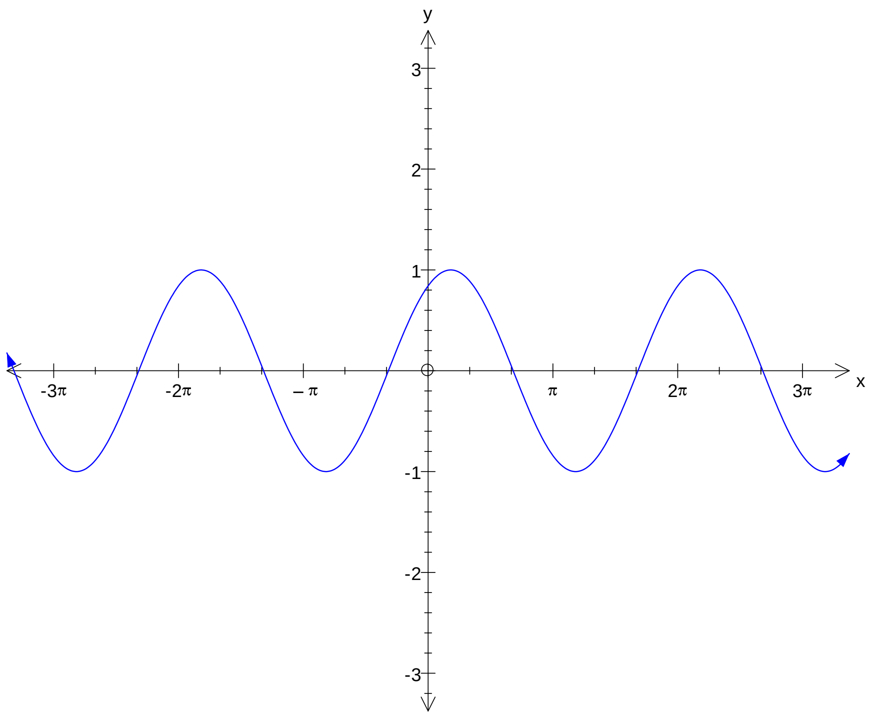
$$(x, y) \to (x+1, y)$$

(x, y)	(x', y')
(x, y)	$(x+1, y)$
Now the image is the $(x+1, y)$	(x', y')
	We will equate the two expression above

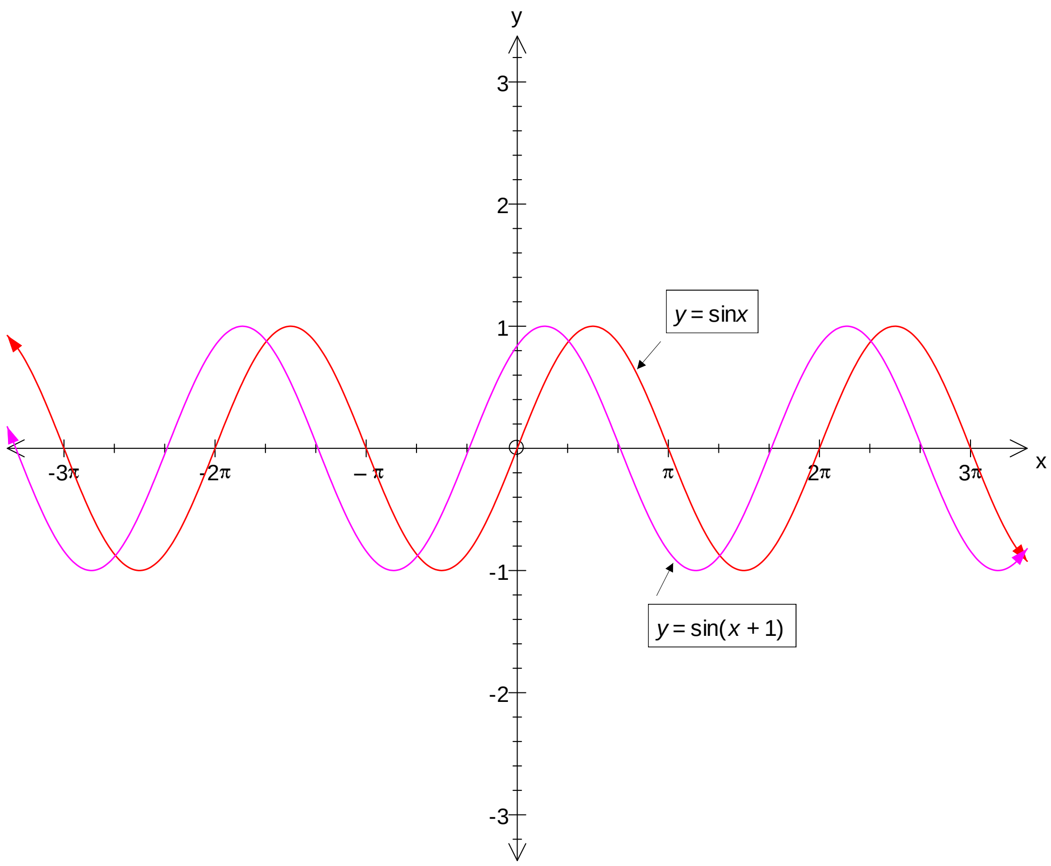
Using the long method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$y' = y$ and $x' = x+1$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x'-1$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \sin(x'-1)$
Now we drop the dashes and we have	$y = \sin(x-1)$

$$y = \sin(x-1)$$

<table>
<tr>
<td>What does the graph look like?</td>
<td>


You cannot really tell much difference unless we overlay the original graph over each other to see what has actually happen



So the graph has been moved to the left by 1 unit
To make it easy to understand this transformation let us use another value
Let’s make it easier to see what is actually happening by using a translation of $+\pi$
</td>
</tr>
</table>

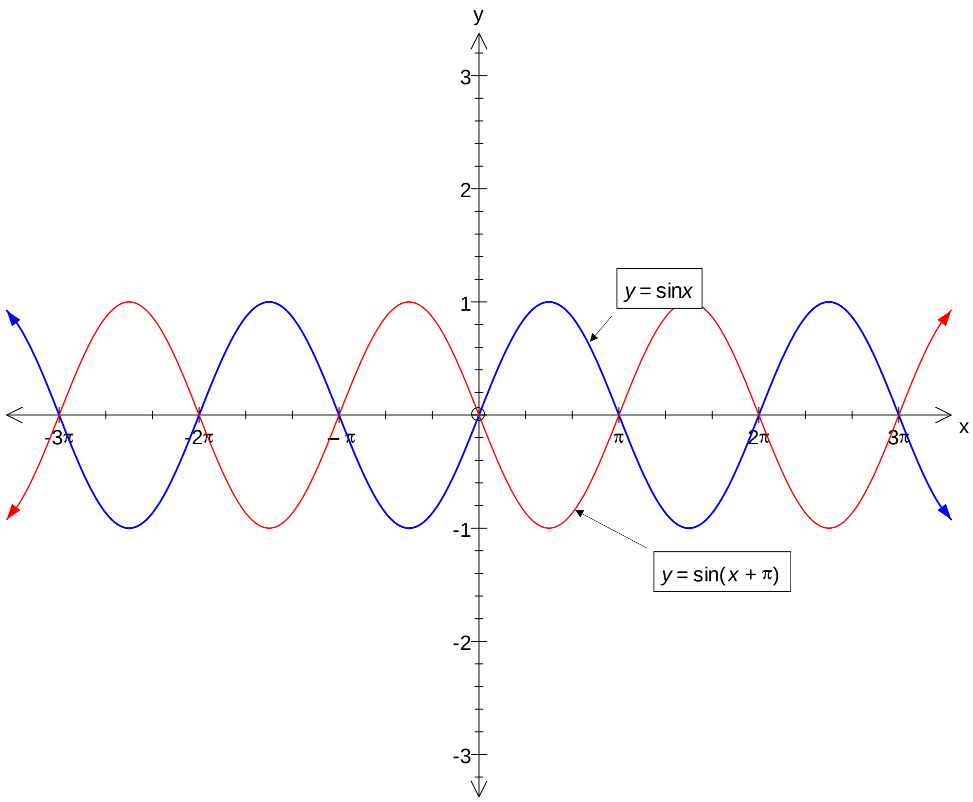
So we go through the entire process again
This time we want to see the effect of translated $+\pi$ unit in the y direction

(x, y)	(x', y')
(x, y)	$(x+\pi, y)$
Now the image is the $(x+\pi, y)$	(x', y')
	We will equate the two expression above

Now using the long method again

What steps we need to do	Steps taken
We equate the dashes with the transformation	$y' = y$ and $x' = x + \pi$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x' - \pi$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \sin(x' - \pi)$
Now we drop the dashes and we have	$y = \sin(x - \pi)$

$y = \sin(x - \pi)$
What does this graph look like?



So the graph has moved to the left by $+\pi$ units!

Reflections

Two types of general reflections
Reflection in the x axis
Reflection in the y axis

And the special reflection in the line

	Reflection in the line $y = x$
Reflection in the x-axis	
	Reflection in the x-axis What is actually happening here $(x, y) \to (x, -y)$ So what happens to the graph

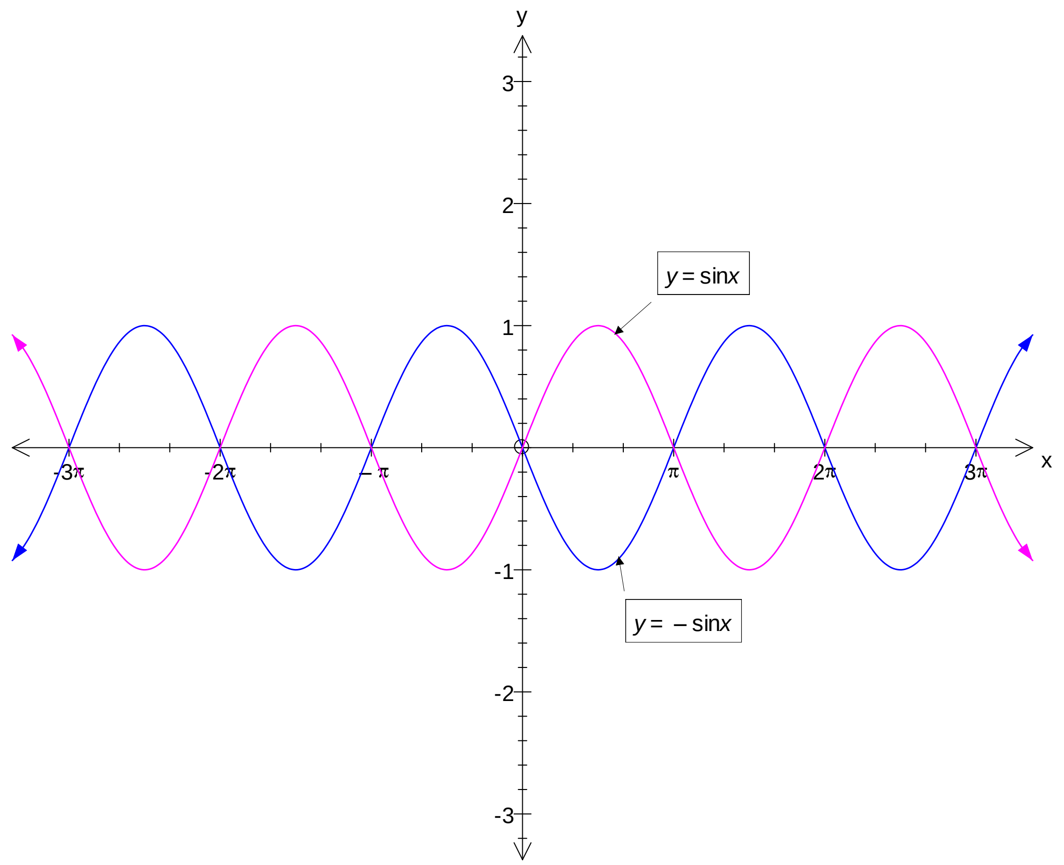
(x, y)	(x', y')
(x, y)	$(x, -y)$
Now the image is the $(x, -y)$	(x', y')
	We will equate the two expression above

Now using the long method again

What steps we need to do	Steps taken
We equate the dashes with the transformation	$y' = -y$ and $x' = x$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x'$ and $y = -y'$
Once we put those expressions into the original equation we get the following	$-y' = \sin x'$
Now we drop the dashes and we have	$-y = \sin x$ $y = -\sin x$

$y = -\sin x$

So what does the graph look like



So the x-axis acts as a mirror line- it is reflected in the x-axis

Reflection in the y-axis

Reflection in the y-axis

What happens here is the following transformation

$(x, y) \to (-x, y)$

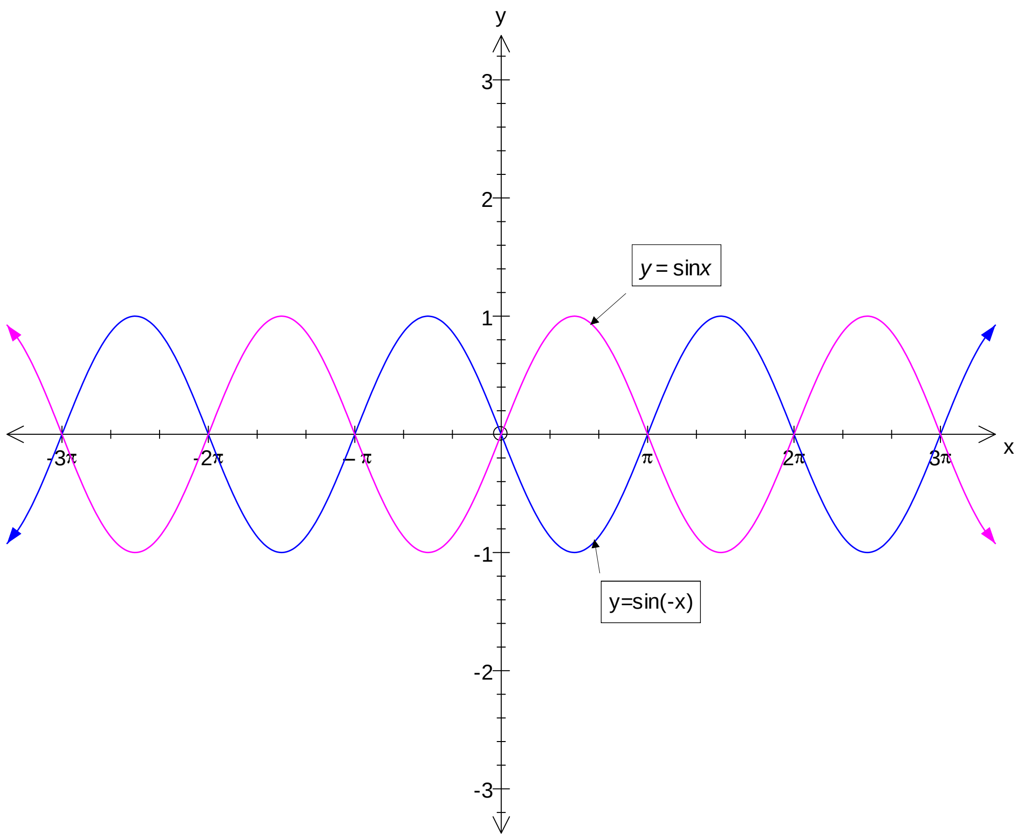
(x, y)	(x', y')
(x, y)	$(-x, y)$
Now the image is the $(-x, y)$	(x', y')
	We will equate the two expression above

Now we use the long method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$y' = y$ and $x' = -x$
Rearrange x and y to be subject so that we can substitute into the original equation	$y' = y$ and $x' = -x$
Once we put those expressions into the original equation we get the following	$y' = \sin(-x')$
Now we drop the dashes and we have	$y = \sin(-x)$

$y = \sin(-x)$

Let's see the graphs now



So the y-axis acts as a mirror axis so it is reflected in the y axis

Reflection along the line $y = x$

Reflection along line $y = x$

Now the reflection is as follows

$(x, y) \to (y, x)$

(x, y)	(x', y')
(x, y)	(y, x)
Now the image is the $(x+1, y)$	(x', y')
	We will equate the two expression above

So comparing to the

$(x, y) \to (x', y')$

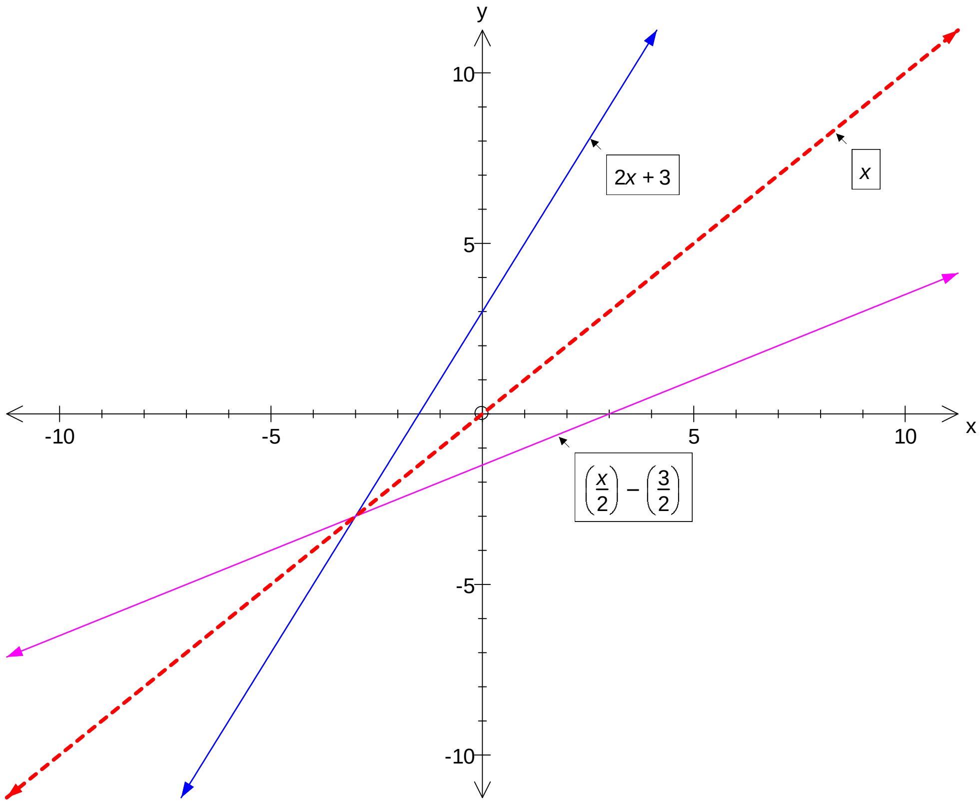
$(x, y) \to (y, x)$

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = y$ and $y' = x$
Rearrange x and y to be subject so that we can substitute into the original equation	$y = x'$ and $x = y'$
Once we put those expressions into the original equation we get the following	$x' = 2y' + 3$ We perform a little rearranging and get $x' - 3 = 2y'$ $\frac{x'}{2} - \frac{3}{2} = y'$ $y' = \frac{x'}{2} - \frac{3}{2}$
Now we drop the dashes and we have	$y = \frac{x}{2} - \frac{3}{2}$

Which becomes $y' = \frac{x'}{2} - \frac{3}{2}$

Now plotting both equations we can see what has happen

$$y = \frac{x}{2} - \frac{3}{2}$$

<table>
<tr><td>How does it look</td><td>

So we can see how the original line is reflected in the dashed line $y = x$</td></tr>
<tr><td colspan="2">Examples to study how they are done- Pay attention to the methods</td></tr>
<tr><td></td><td></td></tr>
<tr><td>Listing the transformation to get to the finish image</td><td>Find the equation of the image $y = \cos x$ under a dilation of factor $\frac{1}{2}$ from the x axis followed by a dilation of factor 3 from the y-axis and then a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

Solution
Start with the original equation</td></tr>
<tr><td>Original equation</td><td>$y = \cos x$</td></tr>
<tr><td>Dilation of factor $\frac{1}{2}$ from the x axis</td><td>Now you could apply the original long algebraic method or you could start to use some of the short methods
We will start to use some of the short methods
Now a dilation of factor k from the x-axis means
This has the following rule $(x, y) \to (x, \frac{1}{2}y)$
So equating the
<table>
<tr><td>(x, y)</td><td>(x', y')</td></tr>
<tr><td>(x, y)</td><td>$(x, \frac{1}{2}y)$</td></tr>
<tr><td>Now the image is the
$(x, y) \to (x, \frac{1}{2}y)$</td><td>$(x', y')$</td></tr>
<tr><td></td><td>We will equate the two expression above</td></tr>
</table></td></tr>
</table>

$x' = x$ and $y' = \frac{1}{2}y$

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = x$ and $y' = \frac{1}{2}y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x'$ and $y = 2y'$
Once we put those expressions into the original equation we get the following	$2y' = \cos x'$ $y' = \frac{\cos x'}{2}$
Now we drop the dashes and we have	$y = \frac{1}{2}\cos x$

Which is basically

$$y = \frac{1}{2}\cos x$$

Dilation of factor 3 from the y axis

The rule for this is

$(x, y) \to (3x, y)$

So equating the

(x, y)	(x', y')
(x, y)	$(3x, y)$
Now the image is the $(x, y) \to (3x, y)$	(x', y')
	We will equate the two expression above

Following the method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = 3x$ and $y' = y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = \frac{x'}{3}$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \frac{1}{2}\cos\left(\frac{x'}{3}\right)$
Now we drop the dashes and we have	$y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

Now the equation becomes $y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

<table>
<tr><td></td><td></td></tr>
<tr><td>translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis</td><td>
What does this mean?
The rule is
The transformation is now
$(x, y) \rightarrow (x+\frac{\pi}{4}, y)$
So comparing to the
$(x, y) \rightarrow (x', y')$
$(x, y) \rightarrow (x+\frac{\pi}{4}, y)$

So equating the
<table>
<tr><td>(x, y)</td><td>(x', y')</td></tr>
<tr><td>(x, y)</td><td>$(x+\frac{\pi}{4}, y)$</td></tr>
<tr><td>Now the image is the
$(x, y) \rightarrow (x+\frac{\pi}{4}, y)$</td><td>$(x', y')$</td></tr>
<tr><td></td><td>We will equate the two expression above</td></tr>
</table>
Following the method again
<table>
<tr><td>What steps we need to do</td><td>Steps taken</td></tr>
<tr><td>We equate the dashes with the transformation</td><td>$x' = x+\frac{\pi}{4}$ and $y' = y$</td></tr>
<tr><td>Rearrange x and y to be subject so that we can substitute into the original equation</td><td>$x = x'-\frac{\pi}{4}$ and $y = y'$</td></tr>
<tr><td>Once we put those expressions into the original equation we get the following</td><td>$y' = \frac{1}{2}\cos\left(\frac{x-\frac{\pi}{4}}{3}\right)$</td></tr>
<tr><td>Now we drop the dashes and we have</td><td>$y' = \frac{1}{2}\cos\left(\frac{1}{3}(x-\frac{\pi}{4})\right)$</td></tr>
</table>

We can write this as $y' = \frac{1}{2}\cos\left(\frac{1}{3}(x-\frac{\pi}{4})\right)$

It takes time but with a little bit of practice you can write the transformation in finished format without having to do the long work!
IF YOU DO NOT FEEL CONFIDENT JUST DO IT THE LONG WAY!!
</td></tr>
<tr><td>Another example</td><td>
Find the image when the graph of $y = \frac{1}{x^2}$ is transformed under the following transformations a dilation of factor 4 from the x-axis

Solution
Always list down the transformation and set up a table it makes life easier in my opinion
This has the following rule $(x, y) \rightarrow (x, 4y)$
So equating the
</td></tr>
</table>

(x, y)	(x', y')
(x, y)	$(x, 4y)$
Now the image is the $(x, y) \to (x, 4y)$	(x', y')
	We will equate the two expression above

Following the method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = x$ and $y' = 4y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x'$ and $y = \frac{y'}{4}$
Once we put those expressions into the original equation we get the following	$\frac{y'}{4} = \frac{1}{(x')^2}$ $y' = \frac{4}{(x')^2}$
Now we drop the dashes and we have	$y = \frac{4}{(x)^2}$

HINT

Here is a summary that might be able to help you do these questions quickly-VERTICAL TRANSFORMATIONS

We start with the following function $f(x)$

Transformation	Meaning	What happens
Reflection in x-axis	Reflected in the x axis	$-f(x)$
Dilation of factor *a* from x axis	Vertical stretch by a factor *a*	$af(x)$
Translation of a units in the positive y direction	Vertical shift by a	$f(x)+a$
Translation of a units in the negative y direction	Vertical shift by a	$f(x)-a$

Now these I call HORIZONTAL TRANSFORMATIONS

Transformation	Meaning	What happens
Reflection in y-axis	Reflected in the y axis	$f(-x)$
Dilation of factor *a* from y axis	Horizontal shrink by a factor *a*	$f\left(\frac{x}{a}\right)$
Translation of a units in the positive x direction	Horizontal shift by a to the left	$f(x-a)$
Translation of a units in the negative x direction	Horizontal shift by a to the right	$f(x+a)$

Remember if you do not feel confident then I strongly suggest that you use the long algebraic method to work the finished transformation equation rather than mess up with the above short cuts.

Also remember the following

Transformation	Meaning
Dilation of factor 2 from x-axis	It will stretch to twice height for y If the dilation factor is >1 then it will stretch
Dilation of $\frac{1}{2}$ from the x- axis	It will shrink to half of its horizontal height If the dilation factor is < 1 it will shrink
Dilation of 2 from y-axis	It will stretch to twice height for x If the dilation factor is >1 then it will stretch
Dilation of $\frac{1}{2}$ from the y- axis	It will shrink to half of its vertical height If the dilation factor is < 1 it will shrink

Practice some of the above techniques to see how quickly you can do them

Exercise-1	Dilation of factor 5 from the y-axis of the equation $y = \sin x$
Exercise-2	Dilation of factor $\frac{1}{2}$ from the y-axis of the equation $y = \sin x$
Exercise-3	Dilation of factor 3 from the x-axis of the equation $y = \sin x$
Exercise-4	Dilation of factor $\frac{1}{7}$ from the x-axis of the equation $y = \sin x$
Exercise-5	Translation of 2 units in the positive y-direction of the equation $y = \sin x$
Exercise-6	Translation of 5 units in the negative y-direction of the equation $y = \sin x$
Exercise-7	Translation of 3 units in the positive x-direction of the equation $y = \sin x$
Exercise-8	Translation of 4 units in the negative x –direction of the equation $y = \sin x$
Exercise-9	Reflection in the x- axis of the equation $y = \sin x$
Exercise-10	Reflection in the y-axis of the equation $y = \sin x$

Answers in order

Let me do the first exercise and see if you can follow the quick method

Transformation	Meaning
Dilation of factor *a* from y axis	$f\left(\frac{x}{a}\right)$
Dilation of factor 5 from the y-axis	$f\left(\frac{x}{5}\right)$ which basically becomes $y = \sin\left(\frac{x}{5}\right)$

Now the long way goes something like this

Always list down the transformation and set up a table it makes life easier in my opinion

This has the following rule $(x, y) \to (5x, y)$

So equating the

(x, y)	(x', y')
(x, y)	$(5x, y)$
Now the image is the $(x, y) \to (5x, y)$	(x', y')
	We will equate the two expression above

Following the method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = 5x$ and $y' = y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = \frac{x'}{5}$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \sin\left(\frac{x'}{5}\right)$
Now we drop the dashes and we have	$y = \sin\left(\frac{x}{5}\right)$

And dropping the dashes gives us

$$y = \sin\left(\frac{x}{5}\right)$$

So can you see that using the short way gets the job done quickly? But it is important to know how to do things the proper long way in the first place.

Answers to the rest	$y = \sin(2x)$, $y = 3\sin(x)$, $y = \frac{1}{7}\sin(x)$, $y = \sin(x) + 2$, $y = \sin(x) - 5$, $y = \sin(x-3)$, $y = \sin(x+4)$, $y = -\sin(x)$, $y = \sin(-x)$
Problems to do	
	Find the equation of the image $y = \sin x$ under the following transformations; a dilation of factor 2 from the x-axis followed by a dilation of factor 3 from the y axis and then a translation of π units in the positive direction of the x-axis
Example	Built the following equation $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$
Solution	Built up the equation starting from the inside first
$y = \sin 2x$	To get this we will need to apply a dilation of factor $\frac{1}{2}$ from the y-axis
$y = \sin 2\left(x - \frac{\pi}{4}\right)$	Now we will try to get the $\frac{\pi}{4}$ which means a translation has occurred So we apply a translation of $\frac{\pi}{4}$ units in the positive x direction. That will give us what we need
$y = 3\sin 2\left(x - \frac{\pi}{4}\right)$	Now a dilation of factor 3 from the x-axis will give us the required transformation
Another Method to do transformation especially when you need to find the final transformed equations	
Important Point	Sometimes especially when a few transformations are taking place we need to be extra careful and *even if we use the short cut we can easily make a mistake*. It is here we can use another interesting method
Problem	
	Consider the equation $y = \sqrt{x}$ And imagine the following transformation are applied to it in the following order: a) Translated 6 units in the negative direction of the x axis b) Reflected in the y-axis

c) Dilated by a factor of 2 from the x-axis

Solution-1

Let use the original long method in doing this problem

Apply transformation 1

Translated 6 units in the negative direction of the x- axis means

The transformation is now

$(x, y) \to (x-6, y)$

So drawing up our little table gives us the following

(x, y)	(x', y')
(x, y)	$(x-6, y)$
Now the image is the $(x, y) \to (x-6, y)$	(x', y')
	We will equate the two expression above

Following the method

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = x - 6$ and $y' = y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = x' + 6$ and $y = y'$
Once we put those expressions into the original equation we get the following	$y' = \sqrt{(x'+6)}$
Now we drop the dashes and we have	$y = \sqrt{(x+6)}$

Next transformation

Now we reflect in the y-axis

This is tricky

What this means is the following

The transformation is now

$(x, y) \to (-x, y)$

So drawing up our little table gives us the following

(x, y)	(x', y')
(x, y)	$(-x, y)$
Now the image is the $(x, y) \to (-x, y)$	(x', y')
	We will equate the two expression above

Put into the previous transformed equation

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = -x$ and $y' = y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = -x'$ and $y = y'$
Once we put those expressions into the previous equation we get the following	$y' = \sqrt{(-x'+6)}$
Now we drop the dashes and we have	$y = \sqrt{(6-x)}$

Which can be written as

$y=\sqrt{(6-x)}$

This would have caused many to make a mistake if they had used the short cut method since most students would had simply put the entire brackets as a negative

Next transformation

Dilation by a factor of 2 from the x-axis

What this means is the following

The transformation is now

$(x,y)\to(x,2y)$

So drawing up our little table gives us the following

(x,y)	(x',y')
(x,y)	$(x,2y)$
Now the image is the $(x,y)\to(x,2y)$	(x',y')
	We will equate the two expression above

$x'=x$ and $y'=2y$

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x'=x$ and $y'=2y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x=x'$ and $y=\dfrac{y'}{2}$
Once we put those expressions into the previous equation we get the following	$\dfrac{y'}{2}=\sqrt{(6-x')}$
Now we drop the dashes and we have	$\dfrac{y}{2}=\sqrt{(6-x)}$ $y=2\sqrt{(6-x)}$

Short cut method from before

Transformation 1

Translated 6 units in the negative direction of the x axis

Translation of a units in the negative x direction	Horizontal shift by a to the right	$f(x+a)$
		$f(x+6)$

Now here we need to be very careful as we are replacing the x with $x+6$

And so we obtain the transformed equation of $y=\sqrt{(x+6)}$

Transformation 2

Reflected in the y axis

Here we would probably make a mistake if we are not careful

The short method says the following

Reflection in y-axis	Reflected in the y axis	$f(-x)$
	$f(-x)$ need to replace x with negative	

So $y=\sqrt{(-x+6)}$ and we have obtained the result

Transformation 3

Dilated by a factor of 2 from the x-axis

Dilation of factor *a* from x axis	Vertical stretch by a factor *a*	$af(x)$

	$2f(x)$	

Using the short cut $y = 2\sqrt{(-x+6)}$

So what is the problem	This method runs into problems and sometimes it is better always to go back to first principles especially when we are having one transformation after another.

Here is another way of doing transformations

Instead of applying the transformation one at a time we could keep using the transformed equation and do the other transformation. Notice what I mean:

Transformation	Starting with	Ending with
Translated 6 units in the negative direction of the x-axis $(x,y) \to (x-6,y)$	(x,y)	$(x-6,y)$
Reflected in the y-axis $(x,y) \to (-x,y)$	$(x-6,y)$ Remember to treat entire $x-6$ as x	$(-(x-6),y)$ Which gives $(6-x,y)$
Dilated by a factor of 2 from x-axis $(x,y) \to (x,2y)$	$(6-x,y)$	$(6-x,2y)$

This is a nice method since we now can proceed with our normal method quickly

(x,y)	(x',y')
(x,y)	$(6-x,2y)$

We know

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x' = 6-x$ and $y' = 2y$
Rearrange x and y to be subject so that we can substitute into the original equation	$x = 6-x'$ and $y = \frac{y'}{2}$
Once we put those expressions into the previous equation we get the following	$\frac{y'}{2} = \sqrt{(6-x')}$
Now we drop the dashes and we have	$\frac{y}{2} = \sqrt{(6-x)}$ $y = 2\sqrt{(6-x)}$

Starting with the $y = x^2$ the following transformations take place

a) Translated 1 unit in the positive direction of the x-axis
b) Translated 2 units in the positive y direction
c) Dilation of factor 2 from y-axis
d) Reflection in the x axis

Solution

Transformation	Starting with	Ending with
Translated 1 units in the positive direction of the x-axis $(x,y)\to(x+1,y)$	(x,y)	$(x+1,y)$
Translated 2 units in the positive direction of the y-axis $(x,y)\to(x,y+2)$	$(x+1,y)$	$(x+1,y+2)$
Reflection in the x-axis $(x,y)\to(x,-y)$	$(x+1,y+2)$	$(x+1,-(y+2))$ Which becomes $(x+1,-y-2)$

Below

This is a nice method since we now can proceed with our normal method quickly

(x,y)	(x',y')
(x,y)	$(x+1,-y-2)$

We know

What steps we need to do	Steps taken
We equate the dashes with the transformation	$x'=x+1$ and $y'=-y-2$
Rearrange x and y to be subject so that we can substitute into the original equation	$x=x'-1$ and $y=-2-y'$
Once we put those expressions into the previous equation we get the following	$-2-y'=(x'-1)^2$ $y'=-2-(x'-1)^2$
Now we drop the dashes and we have	$y=-2-(x-1)^2$

Problem to do	Starting with the following equation $y=\lvert x\rvert$ the following transformation are applied a) Dilation of factor 2 from the x-axis b) Reflection in the x-axis c) Translation 3 units in the positive direction of the x-axis d) Translation of 4 units in the negative direction of the y-axis Solution; $y=-2\lvert x-3\rvert+8$

MATRICES AND TRANSFORMATIONS

You can also use matrices to find the final transformed equation which is probably the easiest method of them all , and that is the reason I have left it to last.

We can use matrices to find the final image under a given transformation below are some summary

Mapping	Rule	Set up for matrix	Matrix
Reflection in the x-axis	$x'=x$ $y'=-y$	$=x+0y$ $=0x+-y$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection in the y-axis	$x'=-x$ $y'=y$	$=-x+0y$ $=0x+y$	$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Dilation by factor k from the y axis	$x'=kx$ $y'=y$	$=kx+0y$ $=0x+y$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Dilation by factor k from the x axis	$x'=x$ $y'=ky$	$=x+0y$ $=0x+ky$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$
Reflection in the line y=x	$x'=y$ $y'=x$	$=0x+y$ $=x+0y$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Translation defined by a vector $\begin{bmatrix}a\\b\end{bmatrix}$	$x'=x+a$ $y'=y+b$	$x'=x+a$ $y'=y+b$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

Example-1	Find the image of the point (2,3) under a reflection in the x-axis Now we can find the image by applying the transformation $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}$ So the image is (2,-3)
Example-2	Find the image of the point (2,3) under a dilation of factor k from the y-axis Now select the transformation $\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2k\\3\end{bmatrix}$ So the image is (2k,3)
Example-3	A transformation is defined by the following $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$ Find the equation of the image $y=x^2+2x+3$ Solution

What steps we need to do	Steps taken
Set up the equation	$(x,y)\rightarrow(x',y')$

Writing up the equation $TX = X'$	$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$
Multiplying the left side by the inverse $T^{-1}TX = T^{-1}X'$ Notice that I multiplied both sides by the inverse Which gives us the following expression $X = T^{-1}X'$	$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}$
X is equal (use the calculator)	$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x'\\\frac{y'}{2}\end{bmatrix}$
Equate the x and y with the dash	$x = x'$ and $y=\frac{y'}{2}$
Substitute into original equation	$\frac{y'}{2}=(x')^2+2(x')+3$
Drop the dashes	$\frac{y}{2}=(x)^2+2(x)+3$ Rearranging gives us $y=2(x^2+2x+3)$

So this is the method we need to use when tackling these type of questions

Example-4

A transformation is described through the equation $T(X+B)=X'$, where

$T=\begin{bmatrix}0 & 1\\-3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1\\2\end{bmatrix}$

Find the image of the straight line with equation $y=-2x+6$

Solution

What steps we need to do	Steps taken
Set up the equation	$(x,y)\to(x',y')$
Writing up the equation $T(X+B)=X'$ And proceed to set it up with X on one side	$T(X+B)=X'$ $T^{-1}T(X+B)=T^{-1}X'$ $(X+B)=T^{-1}X'$ $X=T^{-1}X'-B$
X is equal $X=T^{-1}X'-B$	$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0 & \frac{-1}{3}\\1 & 0\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}-\begin{bmatrix}-1\\2\end{bmatrix}$
Now use the calculator to do the hard work	$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{-y'}{3}+1\\x'-2\end{bmatrix}$
Now write expressions for x and y	$x=\frac{-y'}{3}+1$ $y=x'-2$
Put those expressions into the original equations to obtain the transformation	$y=-2x+6$

$y=-2x+6$	$x'-2=-2\left(\frac{-y'}{3}+1\right)+6$
Drop the dashes and rearrange the equation, you could use the calculator to do the hard work	$x-2=-2\left(\frac{-y}{3}+1\right)+6$ $x-2=\frac{2y}{3}-2+6$ $x-2=\frac{2y}{3}+4$ $x=\frac{2y}{3}+6$ $2y=3x-18$ $y=\frac{3x}{2}-\frac{18}{2}$

So the final answer is $y=\frac{3x}{2}-\frac{18}{2}$ which of course is $y=\frac{3x}{2}-9$

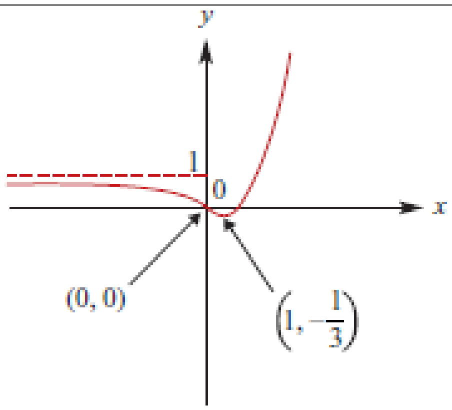
Final note on graphing by looking at transformation

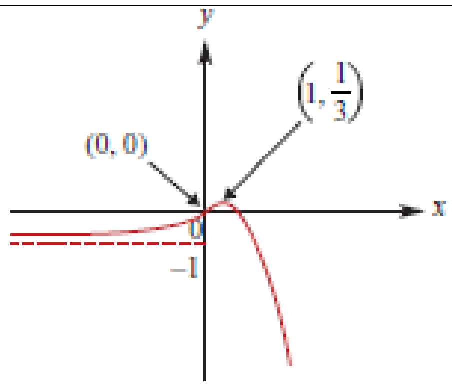
Sometimes we are given a graph and there is no equation to it and we are asked to see what it would be like under certain transformations

Example

Sketch the graph of the image of the graph below under the following sequence of transformations

a) A reflection in the x-axis
b) A dilation of factor 3 from the x-axis
c) A translation of 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y axis

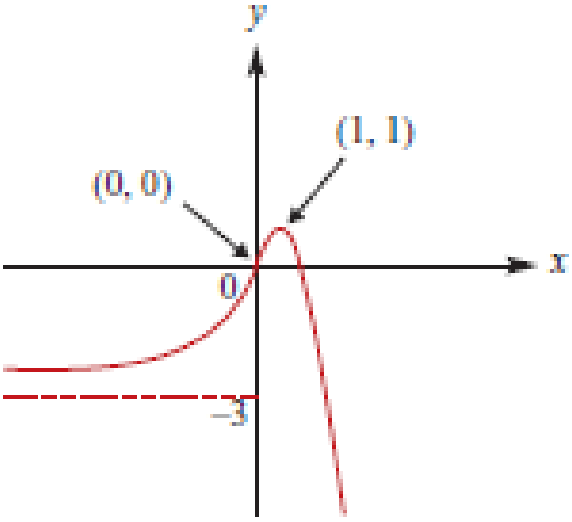




A reflection in the x-axis produces the above graph

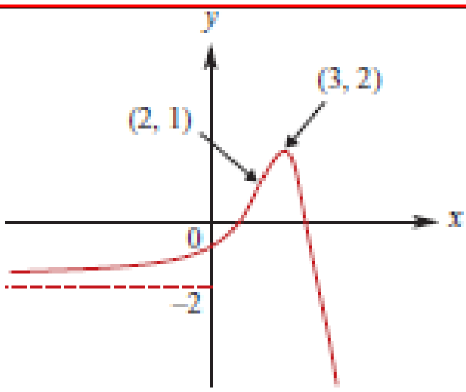
Remember

Transformation	Meaning
Reflection in x-axis	Reflected in the x axis



A dilation of factor 3 from the x-axis- means we multiply by 3 the vertical things

Dilation of factor *a* from x axis	Vertical stretch by a factor *a*	$af(x)$



A translation of 2 units in the positive direction of the x-axis and I unit in the positive direction of the y-axis

Translation of a units in the positive x direction	Horizontal shift by a to the left	$f(x-a)$
Translation of a units in the positive y direction	Vertical shift by a	$f(x)+a$

Transformations-Given the final transformation

How do we find the matrix of the transformation if we are given the beginning and the end of the transformation?

Let us look at an example- 2009 Exam 2 (CAS Q12)

A transformation $T:R^2 \to R^2$ that maps the curve with equation $y=\sin(x)$ onto the curve with equation $y=1-3\sin(2x+\pi)$ is given by what equation below:

A	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ 1 \end{bmatrix}$
B	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 1 \end{bmatrix}$
C	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \pi \\ 1 \end{bmatrix}$
D	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$
E	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{2} \\ -1 \end{bmatrix}$

Start with final image	$y=1-3\sin(2x+\pi)$		
Put the dashes onto the y and x	$y'=1-3\sin(2x'+\pi)$		
Rearrange the equation to have it look like original	$y'=1-3\sin(2x'+\pi)$ $y'-1=-3\sin(2x'+\pi)$ $\frac{y'-1}{-3}=\sin(2x'+\pi)$		
Now set y and x	$y=\frac{y'-1}{-3}$	$x=(2x'+\pi)$	
Express y' in terms y Express x' in terms x	$-3y=y'-1$ And then $y'=-3y+1$	$x-\pi=2x'$ And then $x'=\frac{x}{2}-\frac{\pi}{2}$	
Now we put it into the form of	$\begin{bmatrix} x' \\ y' \end{bmatrix}=T\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} e \\ f \end{bmatrix}$		
this is where you compare the above transformation to see which one fits	$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -\frac{\pi}{2} \\ 1 \end{bmatrix}$ $x'=\frac{x}{2}+0y-\frac{\pi}{2}$ $y'=0x-3y+1$ Answer -D	$x'=\frac{x}{2}-\frac{\pi}{2}$ $y'=-3y+1$	$x'=\frac{x}{2}+0y-\frac{\pi}{2}$ $y'=0x-3y+1$ Compare it with the matrix you will see how the transformation matrix will look like