Chapter 6: Functions, relations and transformations

Exercise 6A Solutions

1 $A = \{1, 2, 3, 5, 7, 11, 15\}$
$B = \{7, 11, 25, 30, 32\}$
$C = \{1, 7, 11, 25, 30\}$
$A \cap B$ means must be in both A and B
$A \cup B$ means must be in either or any
$A \backslash B$ means in A but not B

a $A \cap B = \{7, 11\}$

b $A \cap B \cap C = \{7, 11\}$

c $A \cup B = \{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$

d $A \backslash B = \{1, 2, 3, 5, 15\}$

e $C \backslash B = \{1\}$

f $A \cap C = \{1, 7, 11\}$

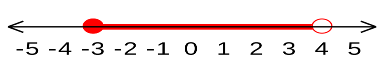
2
a $(-2, 1]$

b $[-3, 3]$

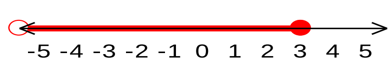
c $(-3, 2)$

d $(-1, 2)$

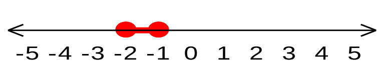
3
a $[-3, 4)$



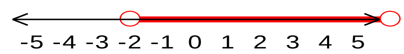
b $(-\infty, 3]$



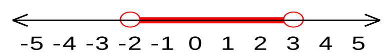
c $[-2, -1]$



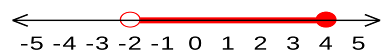
d $(-2, \infty)$



e $(-2, 3)$



f $(-2, 4]$



4
a $\{x: -1 \le x \le 2\} = [-1, 2]$

b $\{x: -4 < x \le 2\} = (-4, 2]$

c $\{y: 0 < y < \sqrt{2}\} = (0, \sqrt{2})$

d $\{y : -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}}\} = (-\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}]$

e $\{x: x > -1\} = (-1, \infty)$

f $\{x: x \le -2\} = (-\infty, -2]$

g $R = (-\infty, \infty)$

h $R^{+} \cup \{0\} = [0, \infty)$

i $R^{-} \cup \{0\} = (-\infty, 0]$

5 $A = \{1, 2, 3, 5, 7, 11, 15\}$
$B = \{7, 11, 25, 30, 32\}$
$C = \{1, 7, 11, 25, 30\}$

a $[-3, 8] \cap C = \{1, 7\}$

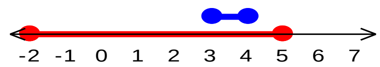
b $(-2, 10] \cap B = \{7\}$

c $(3, \infty) \cap B = \{7, 11, 25, 30, 32\} = B$

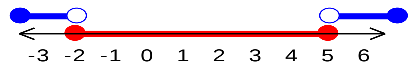
d $(2, \infty) \cup B = (2, \infty)$

6

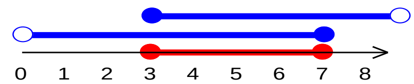
a $[-2, 5]$, $[3, 4]$, $[-2, 5] \cap [3, 4]$



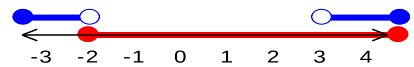
b $[-2, 5]$, $R\backslash[-2, 5]$



c $[3, \infty)$, $(-\infty, 7]$, $[3, \infty) \cap (-\infty, 7]$



d $[-2, 3]$, $R\backslash[-2, 3]$



Exercise 6B Solutions

1

a Domain $[-2, 2]$
range $[-1, 2]$

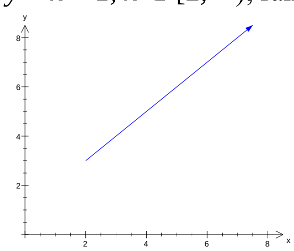
b Domain $[-2, 2]$
range $[-2, 2]$

c Domain R
range $[-1, \infty)$

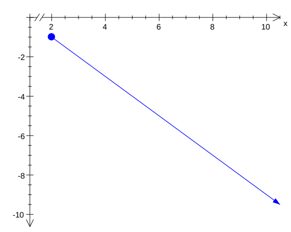
d Domain R
range $(-\infty, 4]$

2

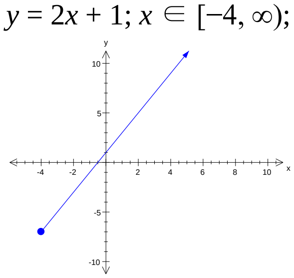
a $y = x + 1;\ x \in [2, \infty)$; range $= [3, \infty)$



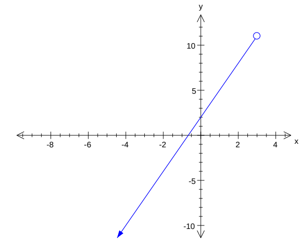
b $y = -x + 1;\ x \in [2, \infty)$; range $= (-\infty, -1]$



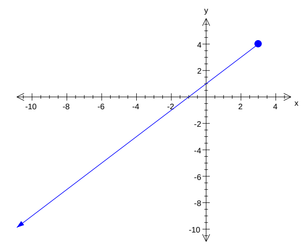
c $y = 2x + 1;\ x \in [-4, \infty)$; range $= [-7, \infty)$



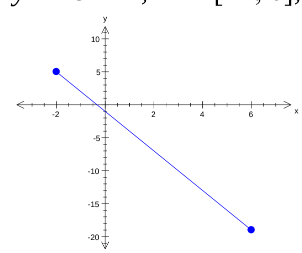
d $y = 3x + 2;\ x \in (-\infty, 3)$; range $= (-\infty, 11)$



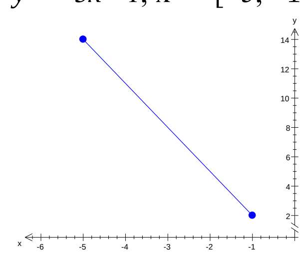
e $y = x + 1;\ x \in (-\infty, 3]$; range $= (-\infty, 4]$



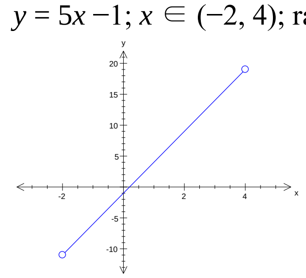
f $y = -3x - 1;\ x \in [-2, 6]$; range $= [-19, 5]$



g $y = -3x - 1;\ x \in [-5, -1]$; range $= [2, 14]$

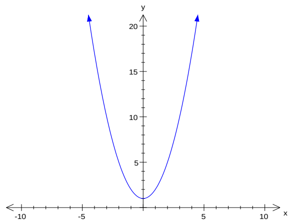


h $y = 5x - 1;\ x \in (-2, 4)$; range $= (-11, 19)$

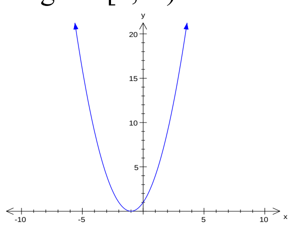


3

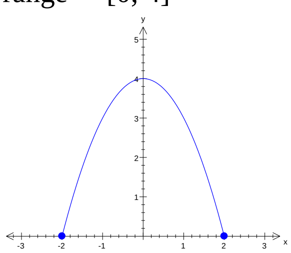
a $\{(x, y): y = x^2 + 1\}$;
range = $[1, \infty)$



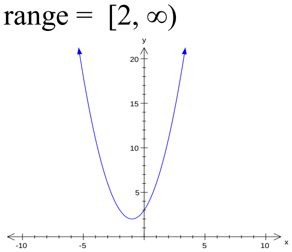
b $\{(x, y): y = x^2 + 2x + 1\}$;
range = $[0, \infty)$



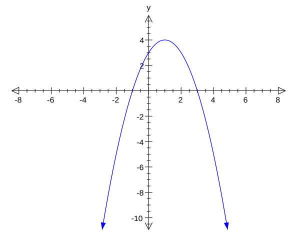
c $\{(x, y): y = 4 - x^2; x \in [-2, 2]\}$;
range = $[0, 4]$



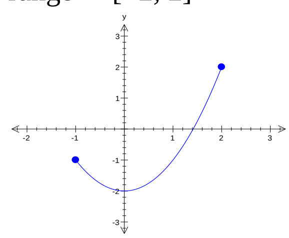
d $\{(x, y): y = x^2 + 2x + 3\}$;
range = $[2, \infty)$



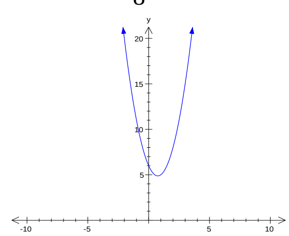
e $\{(x, y): y = -x^2 + 2x + 3\}$;
range = $(-\infty, 4]$



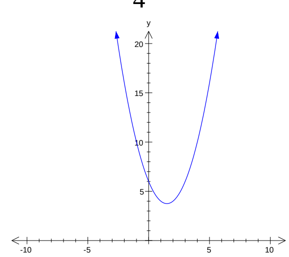
f $\{(x, y): y = x^2 - 2 ; x \in [-1, 2]\}$;
range = $[-2, 2]$



g $\{(x, y): y = 2x^2 - 3x + 6\}$;
range = $[\frac{39}{8}, \infty)$

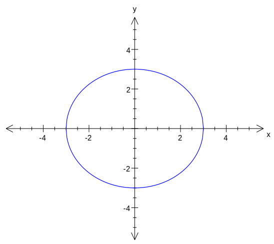


h $\{(x, y): y = 6 - 3x + x^2\}$;
range = $[\frac{15}{4}, \infty)$

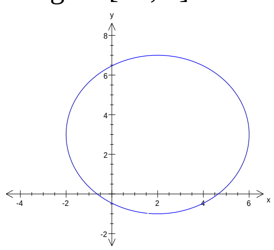


4

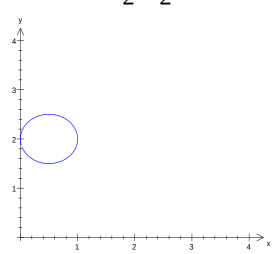
a $\{(x, y): x^2 + y^2 = 9\}$
Max. domain = $[-3, 3]$
Range = $[-3, 3]$



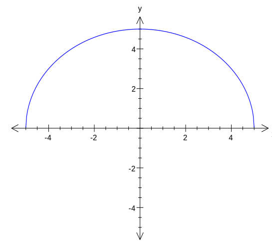
b $\{(x, y): (x - 2)^2 + (y - 3)^2 = 16\}$
Max. domain = $[-2, 6]$
Range = $[-1, 7]$



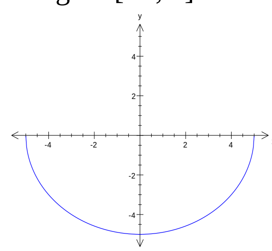
c $\{(x, y): (2x - 1)^2 + (2y - 4)^2 = 1\}$
Max. domain = $[0, 1]$
Range = $[\frac{3}{2}, \frac{5}{2}]$



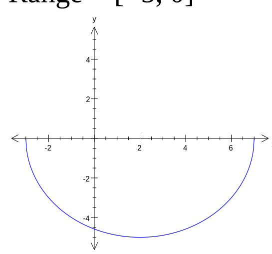
d $\{(x, y): y = \sqrt{25 - x^2}\}$
Max. domain = $[-5, 5]$,
Range = $[0, 5]$



e $\{(x, y): y = -\sqrt{25 - x^2}\}$
Max. domain = $[-5, 5]$,
Range = $[-5, 0]$



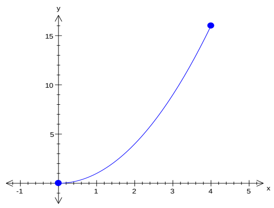
f $(x, y): y = -\sqrt{25 - (x - 2)^2}\}$
Max. domain = $[-3, 7]$
Range = $[-5, 0]$



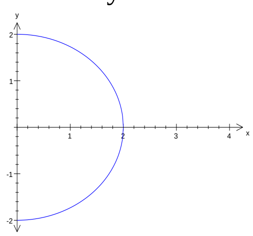
Exercise 6C Solutions

1

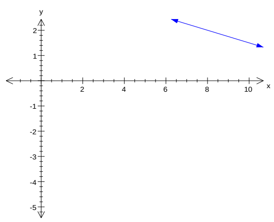
a $y = x^2$; $x \in [0, 4]$; range = $[0, 16]$; function because $1 \longrightarrow 1$ relation



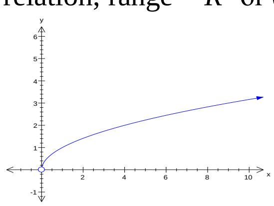
b $\{(x, y)\colon x^2 + y^2 = 4\}$; $x \in [0, 2]$; range = $[-2, 2]$; not a function because $1 \longrightarrow$ many relation



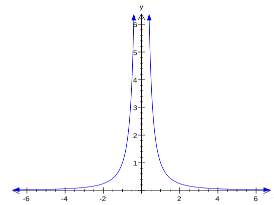
c $\{(x, y)\colon 2x + 8y = 16;\ x \in [0, \infty)\}$; range = $(-\infty, 2]$; function because $1 \longrightarrow 1$ relation



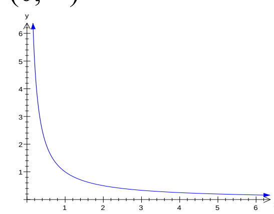
d $y = \sqrt{x}$; $x \in R^+$, function because $1 \longrightarrow 1$ relation; range = R^+ or $(0, \infty)$



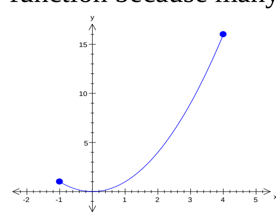
e $\{(x, y)\colon y = \dfrac{1}{x^2};\ x \in R\backslash\{0\}\}$; function because many $\longrightarrow$ 1 relation; range = $R+$ or $(0, \infty)$



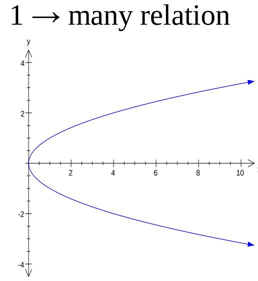
f $\{(x, y)\colon y = \frac{1}{x};\ x \in R^+\}$; function because $1 \longrightarrow 1$ relation; range = R^+ or $(0, \infty)$



g $y = x^2$; $x \in [-1, 4]$; range = $[0, 16]$; function because many $\longrightarrow$ 1 relation



h $\{(x, y)\colon x = y^2;\ x \in R^+\}$; range $= R\backslash\{0\}$; not a function because $1 \longrightarrow$ many relation



2

a $\{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4)\}$ is not a function because it is $1 \to$ many; $(0,1)$ and $(0,2)$ domain = $\{0, 1, 2, 3\}$, range = $\{1, 2, 3, 4\}$

b $\{(-2, -1), (-1, -2), (0, 2), (1, 4), (2, -5)\}$ is a function because it is $1 \to 1$; domain = $\{-2, -1, 0, 1, 2\}$, range = $\{-5, -2, -1, 2, 4\}$

c $\{(0, 1), (0, 2), (-1, 2), (3, 4), (5, 6)\}$ is not a function because it is many $\to$ many; $(0,1)$ and $(0,2)$ domain = $\{-1, 0, 3, 5\}$, range = $\{1, 2, 4, 6\}$

d $\{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$ is a function because it is many $\to 1$; domain = $\{1, 2, 4, 5, 6\}$, range = $\{3\}$

e $\{(x, -2): x \in R\}$ is a function because it is many $\to 1$; domain = R, range = $\{-2\}$

f $\{(3, y): y \in Z\}$ is not a function because it is $1 \to$ many; domain = $\{3\}$, range = Z

g $y = -x + 3$ is a function because it is $1 \to 1$; domain = R, range = R

h $y = x^2 + 5$ is a function because it is many $\to 1$; domain = R, range = $[5, \infty)$

i $\{(x, y): x^2 + y^2 = 9\}$ is not a function because it is many $\to$ many; domain = $[-3, 3]$, range = $[-3, 3]$

3

a $f(x) = 2x - 3$

i $f(0) = 2(0) - 3 = -3$

ii $f(4) = 2(4) - 3 = 5$

iii $f(-1) = 2(-1) - 3 = -5$

iv $f(6) = 2(6) - 3 = 9$

b $g(x) = \frac{4}{x}$

i $g(1) = \frac{4}{1} = 4$

ii $g(-1) = \frac{4}{-1} = -4$

iii $g(3) = \frac{4}{3}$

iv $g(2) = \frac{4}{2} = 2$

c $g(x) = (x - 2)^2$

i $g(4) = (4 - 2)^2 = 4$

ii $g(-4) = (-4 - 2)^2 = 36$

iii $g(8) = (8 - 2)^2 = 36$

iv $g(a) = (a - 2)^2$

d $f(x) = 1 - \frac{1}{x}$

i $f(1) = 1 - \frac{1}{1} = 0$

ii $f(1 + a) = 1 - \frac{1}{1 + a}$

$= \frac{1 + a - 1}{1 + a} = \frac{a}{a + 1}$

iii $f(1 - a) = 1 - \frac{1}{1 - a}$

$= \frac{1 - a - 1}{1 - a}$

$= -\frac{a}{1 - a} = \frac{a}{a - 1}$

iv $f(\frac{1}{a}) = 1 - \frac{1}{1/a} = 1 - a$

4

a $f(x) = 5x - 2 = 3$

$\therefore 5x = 5, \therefore x = 1$

b $f(x) = \dfrac{1}{x} = 6$

$\therefore 1 = 6x, \therefore x = \dfrac{1}{6}$

c $f(x) = x^2 = 9$

$\therefore x = \pm\sqrt{9} = \pm 3$

d $f(x) = (x+1)(x-4) = 0$

$\therefore x = -1, 4$

e $f(x) = x^2 - 2x = 3$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x-3)(x+1) = 0$

$\therefore x = -1, 3$

f $f(x) = x^2 - x - 6 = 0$

$\therefore (x-3)(x+2) = 0$

$\therefore x = -2, 3$

5 $g(x) = x^2 + 2x$ and

$h(x) = 2x^3 - x^2 + 6$

a $g(-1) = (-1)^2 + 2(-1) = -1$

$g(2) = (2)^2 + 2(2) = 8$

$g(-2) = (-2)^2 + 2(-2) = 0$

b $h(-1) = 2(-1)^3 - (-1)^2 + 6 = 3$

$h(2) = 2(2)^3 - (2)^2 + 6 = 18$

$h(-2) = 2(-2)^3 - (-2)^2 + 6 = -14$

c **i** $g(-3x) = (-3x)^2 + 2(-3x) = 9x^2 - 6x$

ii $g(x-5) = (x-5)^2 + 2(x-5)$

$= x^2 - 8x + 15$

iii $h(-2x) = 2(-2x)^3 - (-2x)^2 + 6$

$= -16x^3 - 4x^2 + 6$

iv $g(x+2) = (x+2)^2 + 2(x+2)$

$= x^2 + 6x + 8$

v $h(x^2) = 2(x^2)^3 - (x^2)^2 + 6$

$= 2x^6 - x^4 + 6$

6 $f(x) = 2x^2 - 3$

a $f(2) = 2(2)^2 - 3 = 5$

$f(-4) = 2(-4)^2 - 3 = 29$

b The range of f is $[-3, \infty)$

7 $f(x) = 3x + 1$

a The image of 2 = 3(2) + 1 = 7

b The pre-image of 7: $3x + 1 = 7$

so $3x = 6$ and $x = 2$

c $\{x: f(x) = 2x\}$:

$3x + 1 = 2x, \therefore x = -1$

8 $f(x) = 3x^2 + 2$

a The image of 0 = $3(0)^2 + 2 = 2$

b The pre-image(s) of 5:

$3x^2 + 2 = 5$

$\therefore 3x^2 = 3, \therefore x = \pm 1$

c $\{x: f(x) = 11\}$

$\therefore 3x^2 + 2 = 11$

$\therefore 3x^2 = 9$

$\therefore x^2 = 3, \therefore x = \pm\sqrt{3}$

9 $f(x) = 7x + 6$ and $g(x) = 2x + 1$

a $\{x: f(x) = g(x)\}$

$\therefore 7x + 6 = 2x + 1$

$\therefore 5x = -5, \therefore x = -1$

b $\{x: f(x) > g(x)\}$

$\therefore 7x + 6 > 2x + 1$

$\therefore 5x > -5, \therefore x > -1$

c $\{x: f(x) = 0\}$

$\therefore 7x + 6 = 0$

$\therefore 7x = -6, \therefore x = -\dfrac{6}{7}$

10

a $\{(x, y): y = 3x + 2\}$
$= \{f: R \to R, f(x) = 3x + 2\}$

b $\{(x, y): 2y + 3x = 12\}$

$$2y + 3x = 12$$
$$\therefore \quad 2y = 12 - 3x$$
$$\therefore \quad y = 6 - \frac{3x}{2}$$

$\{f: R \to R, f(x) = 6 - \frac{3x}{2}\}$

c $\{(x, y): y = 2x + 3, x \geq 0\}$
$= \{f: R^+ \cup \{0\} \to R, f(x) = 2x + 3\}$

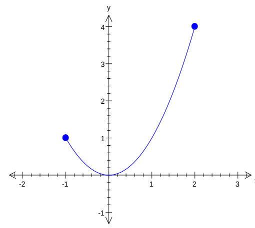
d $y = 5x + 6, -1 \leq x \leq 2$
$\{f: [-1, 2] \to R, f(x) = 5x + 6\}$

e $y + x^2 = 25, -5 \leq x \leq 5$
$\therefore y = 25 - x^2$
$\{f: [-5, 5] \to R, f(x) = 25 - x^2\}$

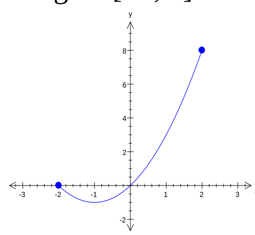
f $y = 5x - 7, 0 \leq x \leq 1$
$\{f: [0, 1] \to R, f(x) = 5x - 7\}$

11

a $f: [-1, 2] \to R, \ f(x) = x^2$
Range = $[0, 4]$

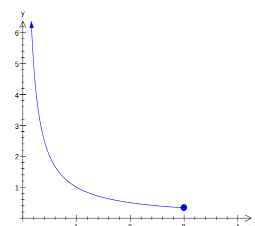


b $f: [-2, 2] \to R, \ f(x) = x^2 + 2x$
Range = $[-1, 8]$

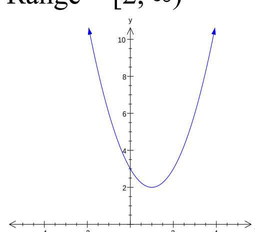


c $f: (0, 3] \to R, f(x) = \frac{1}{x}$

Range = $[\frac{1}{3}, \infty)$



d $f: R \to R, \ f(x) = x^2 - 2x + 3$
Range = $[2, \infty)$



Exercise 6D Solutions

1

a Not 1 ⟶ 1: (2, 4), (4, 4) both have image = 4

b {(1, 3), (2, 4), (3, 6), (7, 9)} is 1⟶ 1

c Not 1 ⟶ 1: $(-x)^2 = x^2$

d {(x, y): $y = 3x + 1$} is 1 ⟶ 1

e $f(x) = x^3 + 1$ is 1 ⟶ 1

f Not 1 ⟶ 1: $1 - x^2 = 1 - (-x)^2$

g $y = x^2$, $x \geq 0$ is 1 ⟶ 1 because $x \geq 0$

2

a Is a function but isn't 1 ⟶ 1 (many ⟶ 1)

b Isn't a function: 1 ⟶ many

c Is a 1 ⟶ 1 function

d Is a function but isn't 1 ⟶ 1 (many ⟶ 1)

e Isn't a function: 1 ⟶ many

f Is a function but isn't 1 ⟶ 1 (many ⟶ 1)

g Is a 1 ⟶1 function

h Isn't a function: many ⟶ many

3

a $y = 7 - x$,
max. domain R, range R

b $y = 2\sqrt{x}$,
max. domain [0, ∞), range [0, ∞)

c $y = x^2 + 1$,
max. domain R , range [1, ∞)

d $y = -\sqrt{9 - x^2}$,
max. domain [−3, 3] because $9 - x^2 \geq 0$,
range [−3, 0]

e $y = \frac{1}{\sqrt{x}}$,
max. domain R^+, range R^+
(Different from **b** because you can't have $\frac{1}{0}$.)

f $y = 3 - 2x^2$,
max. domain R , range (−∞, 3]

g $y = \sqrt{x - 2}$,
max. domain [2, ∞) because $x - 2 \geq 0$,
range [0, ∞)

h $y = \sqrt{2x - 1}$,
max. domain [$\frac{1}{2}$, ∞) because $2x - 1 \geq 0$,
range [0, ∞)

i $y = \sqrt{3 - 2x}$,
max. domain (−∞, $\frac{3}{2}$] because $3 - 2x \geq 0$,
range [0, ∞)

j $y = \frac{1}{2x - 1}$,
max. domain $R\backslash\{\frac{1}{2}\}$ because $2x - 1 \neq 0$,
range $V/\{0\}$ because $\frac{1}{2x - 1} \neq 0$

k $y = \frac{1}{(2x - 1)^2} - 3$,
max. domain $R\backslash\{\frac{1}{2}\}$ because $2x - 1 \neq 0$,
range (−3, ∞) because $\frac{1}{(2x - 1)^2} > 0$

l $y = \frac{1}{2x - 1} + 2$,
max. domain $R\backslash\{\frac{1}{2}\}$ because $2x - 1 \neq 0$,
range $V/\{2\}$ because $\frac{1}{2x - 1} \neq 0$

4

a $f(x) = 3x + 4$;
max. domain R, range R

b $g(x) = x^2 + 2$,
max. domain R, range $[2, \infty)$

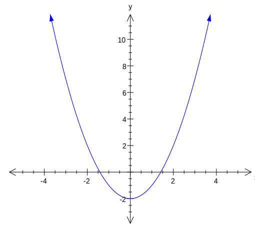
c $y = -\sqrt{16 - x^2}$,
max. domain $[-4, 4]$ because $16 - x^2 \geq 0$,
range $[-4, 0]$

d $y = \dfrac{1}{x + 2}$,
max domain $R\backslash\{-2\}$ because $x + 2 \neq 0$,
range $R\backslash\{0\}$ because $\dfrac{1}{x + 2} \neq 0$

5 $\{(x, y)\colon y^2 = -x + 2, x \leq 2\}$ is a one $\to$ many relation. Split in two:
$\{f\colon (-\infty, 2],\ f(x) = \sqrt{2 - x}\}$, range $[0, \infty)$
$\{f\colon (-\infty, 2],\ f(x) = -\sqrt{2 - x}\}$, range $(-\infty, 0]$

6

a $\{f: R \to R;\ f(x) = x^2 - 2\}$

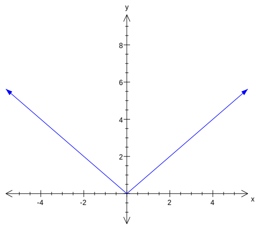


b $\{f\colon [0, \infty) \to R;\ f(x) = x^2 - 2\}$ and
$\{f\colon (-\infty, 0] \to R;\ f(x) = x^2 - 2\}$

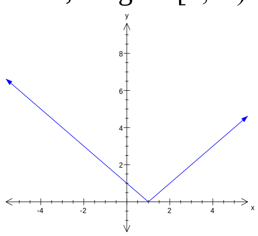
Exercise 6E Solutions

1

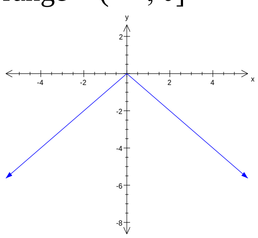
a $h(x) = x,\ x \geq 0$ and $h(x) = -x,\ x < 0$; range = $[0, \infty)$



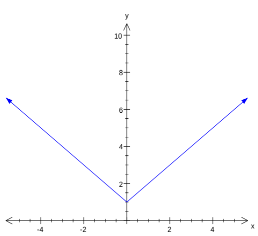
b $h(x) = x - 1,\ x \geq 1$ and $h(x) = 1 - x,\ x < 1$; range = $[0, \infty)$



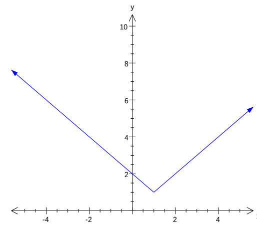
c $h(x) = -x,\ x \geq 0$ and $h(x) = x,\ x < 0$; range = $(-\infty, 0]$



d $h(x) = 1 + x,\ x \geq 0$ and $h(x) = 1 - x,\ x < 0$; range = $[1, \infty)$



e $h(x) = x,\ x \geq 1$ and $h(x) = 2 - x,\ x < 1$; range = $[1, \infty)$



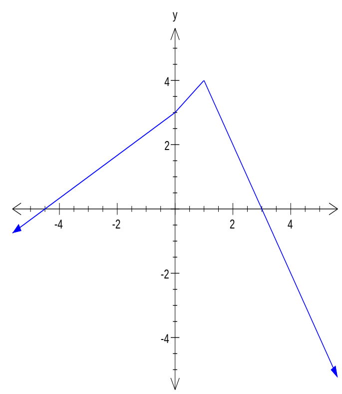
2

a $f(x) = \frac{2}{3}x + 3,\ x < 0$

$f(x) = x + 3,\ 0 \leq x \leq 1$

$f(x) = -2x + 6,\ x > 1$

Axis intercepts at $(-\frac{9}{2}, 0)$, $(0,3)$ and $(3,0)$



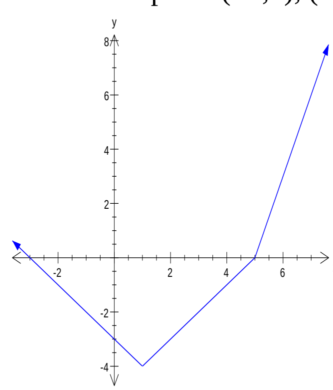
b Range = $(-\infty, 4]$

3 $g(x) = -x - 3,\ x < 1$

$g(x) = x - 5,\ 1 \leq x \leq 5$

$g(x) = 3x - 15,\ x > 5$

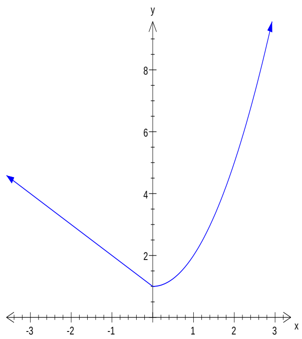
Axis intercepts at $(-3,0)$, $(0,-3)$ and $(5,0)$



4

a $h(x) = x^2 + 1,\ x \geq 0$

$h(x) = 1 - x,\ x < 0$



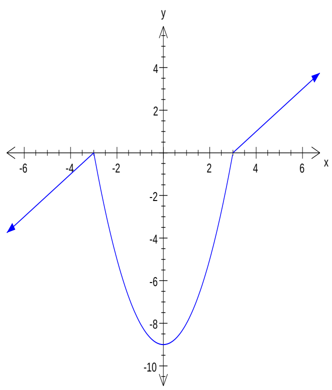
b Range = $[1, \infty)$

5

a $f(x) = -x + 3,\ x < -3$

$f(x) = x^2 - 9,\ -3 \leq x \leq 3$

$f(x) = x - 3,\ x > 3$

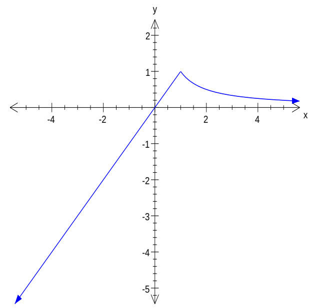


b Range = R

6

a $f(x) = \dfrac{1}{x},\ x > 1$

$f(x) = x,\ x \leq 1$



b Range = $(-\infty, 1]$

7 Line connecting $(-3, 0)$ and $(-1, 2)$ has

gradient $= \dfrac{2 - 0}{-1 - (-3)} = 1$

Using $(-3, 0)$: $y - 0 = 1(x - -3)$

$\therefore y = x + 3$ for $[-3, -1]$

Line connecting $(-1, 2)$ and $(2, -1)$ has

gradient $= \dfrac{-1 - 2}{2 - (-1)} = -1$

Using $(-1, 2)$: $y - 2 = -1(x - -1)$

$\therefore y = 1 - x$ for $[-1, 2]$

Line connecting $(2, -1)$ and $(4, -2)$ has

gradient $= \dfrac{-2 - -1}{4 - 2} = -\dfrac{1}{2}$

Using $(2, -1)$: $y + 1 = -\dfrac{1}{2}(x - 2)$

$\therefore y = -\dfrac{x}{2}$ for $[2, 4]$

$$f(x) = \begin{cases} x + 3; & -3 \leq x < -1 \\ 1 - x; & -1 \leq x < 2 \\ -\dfrac{x}{2}; & 2 \leq x \leq 4 \end{cases}$$

Exercise 6F Solutions

1

a $f(x) = a + bx$

$f(4) = -1 \quad \therefore a + 4b = -1$

$f(8) = 1 \quad \therefore a + 8b = 1$

$\therefore \quad 4b = 2$

$\therefore b = \frac{1}{2}; a = -3$

b $f(x) = 0, \therefore \frac{x}{2} - 3 = 0$

$\therefore \quad x = 6$

2 If $f(x)$ is parallel to $g(x) = 2 - 5x$ then the gradient of $f(x) = -5$ and

$f(x) = -5x + c$

$f(0) = 7, \therefore c = 7$

$f(x) = -5x + 7$

3 $f(x) = ax + b$

$f(-5) = -12 \quad \therefore -5a + b = -12$

$f(7) = 6 \quad \therefore 7a + b = 6$

$\therefore 12a = 18$

a **i** $f(0) = b = -\frac{9}{2}$

ii $f(1) = \frac{3}{2} - \frac{9}{2} = -3$

b $f(x) = \frac{1}{2}(3x - 9) = 0$

$\therefore 3x - 9 = 0, \therefore x = 3$

4 $f(x) = 2x + 5$

a $f(p) = 2p + 5$

b $f(p + h) = 2p + 2h + 5$

c $f(p + h) - f(p)$

$= (2p + 2h + 5) - (2p + 5) = 2h$

d $f(p + 1) - f(p)$

$= (2p + 2 + 5) - (2p + 5) = 2$

5 $f(x) = 3 - 2x$

$f(p + 1) - f(p)$

$= (3 - 2(p + 1)) - (3 - 2p)$

$= 3 - 2p - 2 - 3 + 2p$

$= -2$

6

a $L(C) = 0.002C + 25$; $-273 \le C \le 1000$

Most metals will melt at over 1000 degrees and $C = -273$ is absolute zero.

b **i** $L(30) = (0.002)30 + 25 = 25.06$ cm

ii $L(16) = (0.002)16 + 25 = 25.032$ cm

iii $L(100) = (0.002)100 + 25= 25.20$ cm

iv $L(500) = (0.002)500 + 25= 26.00$ cm

7 $f(x) = a(x - b)(x - c)$

$f(2) = f(4) = 0$ so $b = 2$, $c = 4$

If 7 is maximum then $a < 0$; max. occurs halfway between 2 and 4, i.e. at $x = 3$:

$f(x) = a(3 - 2)(3 - 4) = 7$

$\therefore a = -7$

$\therefore f(x) = -7(x - 2)(x - 4)$

OR $f(x) = -7x^2 + 42x - 56$

8 $f(x) = x^2 - 6x + 16$

$= x^2 - 6x + 9 + 7$

$= (x - 3)^2 + 7$

Range of $f = [7, \infty)$

9

a $f(x) = -2x^2 + x - 2$

$= -2\left(x^2 - \frac{x}{2} + 1\right)$

$= -2\left(x^2 - \frac{x}{2} + \frac{1}{16} + \frac{15}{16}\right)$

$= -2\left(x - \frac{1}{4}\right)^2 - \frac{15}{8}$

Range of $f = (-\infty, -\frac{15}{8}]$

b $f(x) = 2x^2 - x + 4$

$= 2\left(x^2 - \frac{x}{2} + 2\right)$

$= 2\left(x^2 - \frac{x}{2} + \frac{1}{16} + \frac{31}{16}\right)$

$= 2\left(x - \frac{1}{4}\right)^2 + \frac{31}{8}$

Range of $f = [\frac{31}{8}, \infty)$

c $f(x) = -x^2 + 6x + 11$

$= -(x^2 - 6x - 11)$

$= -(x^2 - 6x + 9 - 20)$

$= -(x - 3)^2 + 20$

Range of $f = (-\infty, 20]$

d $g(x) = -2x^2 + 8x - 5$

$= -2\left(x^2 - 4x + \frac{5}{2}\right)$

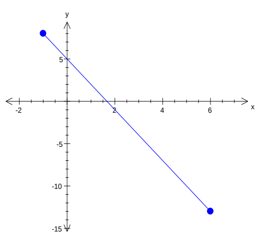
$= -2\left(x^2 - 4x + 4 - \frac{3}{2}\right)$

$= -2(x - 2)^2 + 3$

Range of $g = (-\infty, 3]$

10 f: $[-1, 6] \rightarrow R$, $f(x) = 5 - 3x$

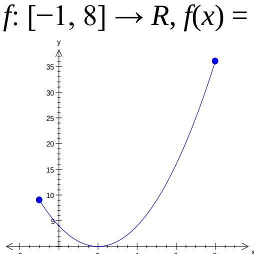
a



b Range of $f = [-13, 8]$

11

a f: $[-1, 8] \rightarrow R$, $f(x) = (x - 2)^2$



b Range of $f = [0, 36]$

12

a $x^2 + y^2 = 9$

Circle: radius 3, centre (0,0)

Implied domain = $[-3, 3]$, range = $[-3, 3]$

b $(x - 2)^2 + y^2 = 1$

Circle: radius 1, centre (2,0)

Implied domain = $[1, 3]$, range = $[-1, 1]$

c $(2x - 1)^2 + (2y - 1)^2 = 1$

Circle: radius $\frac{1}{2}$, centre $(\frac{1}{2}, \frac{1}{2})$

Implied domain = $[0, 1]$, range = $[0, 1]$

d $(x - 4)^2 + y^2 = 25$

Circle: radius 5, centre (4,0)

Implied domain = $[-1, 9]$, range = $[-5, 5]$

e $(y - 2)^2 + x^2 = 16$

Circle: radius 4, centre (0,2)

Implied domain = $[-4, 4]$, range = $[-2, 6]$

13 Domain of the function f is $\{1, 2, 3, 4\}$

a $f(x) = 2x$ so range = $\{2, 4, 6, 8\}$

b $f(x) = 5 - x$ so range = $\{1, 2, 3, 4\}$

c $f(x) = x^2 - 4$ so range = $\{-3, 0, 5, 12\}$

d $f(x) = \sqrt{x}$ so range = $\{1, \sqrt{2}, \sqrt{3}, 2\}$

14 f: $R \rightarrow R$, $f(x) = m(x - p)(x - q)$

$f(4) = f(5) = 0$, so $p = 4$, $q = 5$

$f(0) = 2$, so $mpq = 2$ and $m = 0.1$

$f(x) = 0.1(x - 4)(x - 5)$

$= 0.1(x^2 - 9x + 20)$

$= 0.1x^2 - 0.9x + 2$

$a = 0.1$, $b = -0.9$, $c = 2$

OR Use $f(0) = 2$ so $c = 2$:

$f(4) = 0$, so $16a + 4b + 2 = 0$

$f(5) = 0$, so $25a + 9b + 2 = 0$

and use simultaneous equations or matrices.

15 $f(x) = ax^2 + bx + c$

$f(0) = 10$ so $c = 10$

Max. value = 18 at $x = -\dfrac{b}{2a}$:

$$f\left(-\frac{b}{2a}\right) = 18$$

$$= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + 10$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + 10$$

$$= -\frac{b^2}{4a} + 10$$

$\therefore -\dfrac{b^2}{4a} = 8$

$\therefore b^2 = -32a$ … **(1)**

$f(1) = 0$

$\therefore a + b + 10 = 0$

$\therefore b = -10 - a$

$\therefore b^2 = (10 + a)^2$ … **(2)**

Equate **(1)** and **(2)**:

$\therefore (10 + a)^2 = -32a$

$\therefore a^2 + 20a + 100 = -32a$

$\therefore a^2 + 52a + 100 = 0$

$\therefore (a + 50)(a + 2) = 0$

$\therefore a = -2, -50$

If $a = -2$, $b = -8$; if $a = -50$, $b = 40$

$\therefore f(x) = -2x^2 - 8x + 10$

$g(x) = -50x^2 + 40x + 10$

OR $f(x) = -2(x - 1)(x + 5)$

$g(x) = -10(5x + 1)(x - 1)$

16

a $f(x) = 3x^2 - 5x - k$

$f(x) > 1$ for all real x

So $f(x)$-1> for all real x

$1\,3x^2 - 5x - (k+1) > 0$ for all real x .

Then there are two real solutions to the equation $3x^2 - 5x - (k+1) = 0$, so $\Delta < 0$.

$\therefore 12k < -37$

$\therefore k < -\dfrac{37}{12}$

< 0 if k $< -\dfrac{37}{12}$

b $a > 0$ so the curve is an upright parabola, so the vertex is the minimum value which occurs at $x = -\dfrac{b}{2a}$

For $a = 3$ and $b = -5$, $x = \dfrac{5}{6}$

$$f\left(\frac{5}{6}\right) = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) - k = 0$$

$\therefore \dfrac{25}{12} - \dfrac{25}{6} - k = 0$

$\therefore k = -\dfrac{25}{12}$

Exercise 6G Solutions

1

a Inverse = {(3, 1), (6, −2), (5, 4), (1, 7)} (just swap coordinates).
For f^{-1}, domain = {1,3,5,6};
range = {−2,1,4,7}

b f: $R \to R$, $f(x) = 6 - 2x$ has inverse:

$x = 6 - 2y, \therefore y = 3 - \frac{x}{2}$

$\therefore f^{-1}(x) = 3 - \frac{x}{2}$

Domain = R, range = R

c f: $[1, 5] \to R$, $f(x) = 3 - x$ has inverse:

$x = 3 - y, \therefore y = 3 - x$

$\therefore f^{-1}(x) = 3 - x$

Domain = [−2, 2] (range of f),
range = [1, 5] (domain of f)

d f: $R^+ \to R$, $f(x) = x + 4$ has inverse:

$x = y + 4, \therefore y = x - 4$

$\therefore f^{-1}(x) = x - 4$

Domain = (4, ∞) (range of f),
range = R^+ (domain of f)

e f: $(-\infty, 4] \to R$, $f(x) = x + 4$ has inverse:

$f^{-1}(x) = x - 4$

Domain = (−∞, 8] (range of f),
range = (−∞, 4) (domain of f)

f f: $[0,\infty) \to R$, $f(x) = x^2$

$x = y^2, \therefore y = \sqrt{x}$

$\therefore f^{-1}(x) = \sqrt{x}$

Domain = [0, ∞) (range of f),
range = [0, ∞) (domain of f)

g f: $[2,\infty) \to R$, $f(x) = (x - 2)^2 + 3$

$x = (y - 2)^2 + 3$

$\therefore (y - 2)^2 = x - 3$

$\therefore y - 2 = \sqrt{x - 3}$

$\therefore y = \sqrt{x - 3} + 2$

$\therefore f^{-1}(x) = \sqrt{x - 3} + 2$

Domain = [3, ∞) (range of f),
range = [2, ∞) (domain of f)

h f: $(-\infty, 4] \to R$, $f(x) = (x - 4)^2 + 6$

$x = (y - 4)^2 + 6$

$\therefore x - 6 = (y - 4)^2$

$\therefore y - 4 = -\sqrt{x - 6}$

This time we need the negative square root because of the domain of f, which is restricted to the left-hand side of the graph.

$\therefore y = 4 - \sqrt{x - 6}$

$\therefore f^{-1}(x) = 4 - \sqrt{x - 6}$

Domain = [6, ∞) (range of f),
range = (−∞, 4] (domain of f)

i f: $[0, 1] \to R$, $f(x) = \sqrt{1 - x}$

$x = \sqrt{1 - y}$

$\therefore x^2 = 1 - y$

$\therefore y = 1 - x^2$

$\therefore f^{-1}(x) = 1 - x^2$

Domain = [0, 1] (range of f),
range = [0, 1] (domain of f)

j f: $[0, 4] \to R$, $f(x) = \sqrt{16 - x^2}$

$x = \sqrt{16 - y^2}$

$\therefore x^2 = 16 - y^2$

$\therefore y^2 = 16 - x^2$

$\therefore y = \sqrt{16 - x^2}$

$\therefore f^{-1}(x) = \sqrt{16 - x^2}$

Domain = [0, 4] (range of f),
range = [0, 4] (domain of f)

k f: $[-1, 7] \to R$, $f(x) = 16 - 2x$

$x = 16 - 2y$

$\therefore 2y = 16 - x$

$\therefore y = 8 - \frac{x}{2}$

$\therefore f^{-1}(x) = 8 - \frac{x}{2}$

Domain = [2, 18] (range of f),
range = [−1, 7] (domain of f)

1 $f: [0, \infty) \to R, f(x) = (x+4)^2 + 6$

$x = (y+4)^2 + 6$

$\therefore x - 6 = (y+4)^2$

$\therefore y + 4 = \sqrt{x-6}$

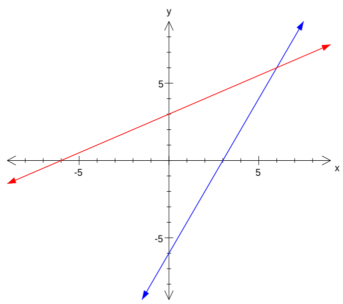
$\therefore y = \sqrt{x-6} - 4$

$\therefore f^{-1}(x) = \sqrt{x-6} - 4$

Domain = $[22, \infty)$ (range of f), range = $[0, \infty)$ (domain of f)

2

a



b $f(x) = f^{-1}(x)$ when $2x - 6 = \frac{x}{2} + 3$

$\therefore \frac{3x}{2} = 9, \therefore x = 6$

When $x = 6$, $y = 6$ so $(6, 6)$

3

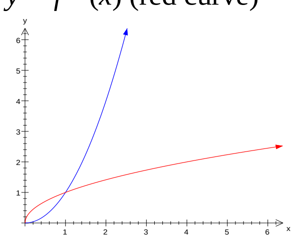
a $f: [0,\infty) \to R, f(x) = x^2$

$\therefore f^{-1}(x) = \sqrt{x}$

Positive roots because domain of f is positive.

$y = f(x)$ (blue curve);

$y = f^{-1}(x)$ (red curve)



b $f(x) = f^{-1}(x)$ where $x^2 = \sqrt{x}$

$\therefore x^4 = x, \therefore x^4 - x = 0$

$\therefore x(x^3 - 1) = 0$

$\therefore x = 0, 1$

i.e. at $(0, 0)$ and $(1, 1)$

4 $f: R \to R, f(x) = ax + b, a, b \neq 0$

$f(1) = 2, \therefore a + b = 2$

$f^{-1}(x) = \frac{x - b}{a}$

$f^{-1}(1) = \frac{1-b}{a} = 3$

$\therefore 1 - b = 3a$

$\therefore 3a + b = 1$

$\underline{\quad a + b = 2}$

$\therefore 2a \quad = -1$

$\therefore a = -\frac{1}{2}; b = \frac{5}{2}$

5 $f: (-\infty, a] \to R, f(x) = \sqrt{a - x}$

a $x = \sqrt{a - y}$

$\therefore x^2 = a - y, \therefore y = a - x^2$

$f^{-1}(x) = a - x^2, x \geq 0$

(to match range of f)

b At $x = 1$: $\sqrt{a - x} = a - x^2$

$\therefore \sqrt{a-1} = a - 1$

$\therefore a - 1 = (a-1)^2$

$\therefore a^2 - 2a + 1 - a + 1 = 0$

$\therefore a^2 - 3a + 2 = 0$

$\therefore (a-2)(a-1) = 0$

$\therefore a = 1, 2$

Exercise 6H Solutions

1

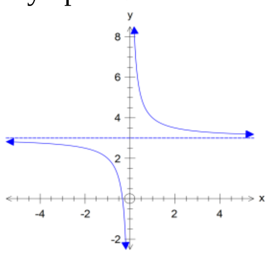
a $y = \frac{1}{x} + 3$

Asymptotes at $x = 0$ and $y = 3$

x-intercept: $y = \frac{1}{x} + 3 = 0$

$\therefore \frac{1}{x} = -3, \ \therefore x = -\frac{1}{3}$

No y-intercept because $x = 0$ is an asymptote.



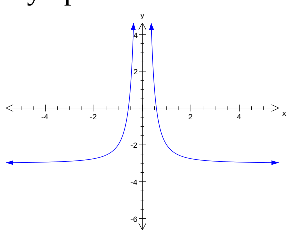
b $y = \frac{1}{x^2} - 3$

Asymptotes at $x = 0$ and $y = -3$

x-intercept: $y = \frac{1}{x^2} - 3 = 0$

$\therefore \frac{1}{x^2} = 3, \ \therefore x = \pm\frac{1}{\sqrt{3}}$

No y-intercept because $x = 0$ is an asymptote.

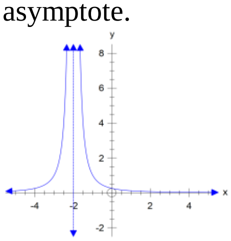


c $y = \frac{1}{(x+2)^2}$

Asymptotes at $x = -2$ and $y = 0$

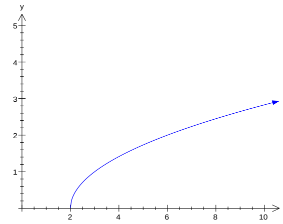
y-intercept: $y = \frac{1}{2^2} - 3 = -\frac{11}{4}$

No x-intercept because $y = 0$ is an asymptote.



d $y = \sqrt{x - 2}$

No asymptotes, starting point at (2,0).

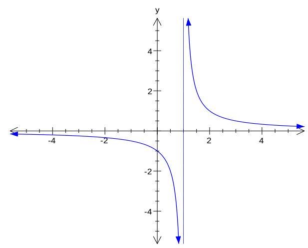


e $y = \frac{1}{x - 1}$

Asymptotes at $x = 1$ and $y = 0$

y-intercept: $y = \frac{1}{0 - 1} = -1$

No x-intercept because $y = 0$ is an asymptote.



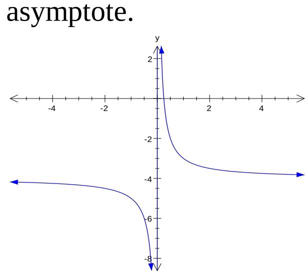
f $y = \frac{1}{x} - 4$

Asymptotes at $x = 0$ and $y = -4$

x-intercept: $y = \frac{1}{x} - 4 = 0$

$\therefore \frac{1}{x} = 4, \ \therefore x = \frac{1}{4}$

No y-intercept because $x = 0$ is an asymptote.

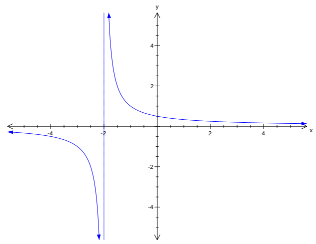


g $y = \dfrac{1}{x+2}$

Asymptotes at $x = -2$ and $y = 0$

y-intercept: $y = \dfrac{1}{0+2} = \dfrac{1}{2}$

No x-intercept because $y = 0$ is an asymptote.

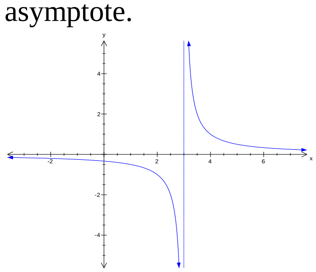


h $y = \dfrac{1}{x-3}$

Asymptotes at $x = 3$ and $y = 0$

y-intercept: $y = \dfrac{1}{0-3} = -\dfrac{1}{3}$

No x-intercept because $y = 0$ is an asymptote.

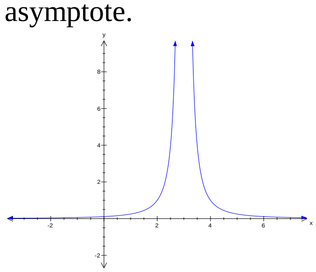


i $f(x) = \dfrac{1}{(x-3)^2}$

Asymptotes at $x = 3$ and $y = f(x) = 0$

y-intercept: $f(0) = \dfrac{1}{(0-3)^2} = \dfrac{1}{9}$

No x-intercept because $y = 0$ is an asymptote.

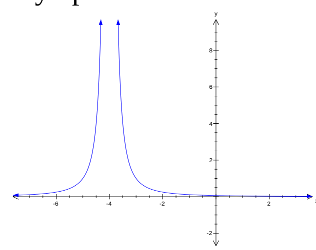


j $f(x) = \dfrac{1}{(x+4)^2}$

Asymptotes at $x = -4$ and $y = f(x) = 0$

y-intercept: $f(0) = \dfrac{1}{(0+4)^2} = \dfrac{1}{16}$

No x-intercept because $y = 0$ is an asymptote.



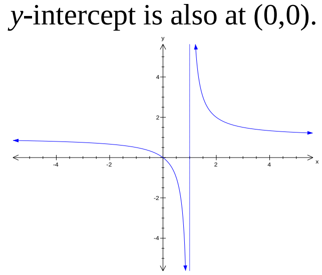
k $f(x) = \dfrac{1}{x-1} + 1$

Asymptotes at $x = 1$ and $y = f(x) = 1$

x-intercept: $y = \dfrac{1}{x-1} + 1 = 0$

$\therefore \dfrac{1}{x-1} = -1, \ \therefore x = 0$

y-intercept is also at (0,0).



l $f(x) = \dfrac{1}{x-2} + 2$

Asymptotes at $x = 2$ and $y = f(x) = 2$

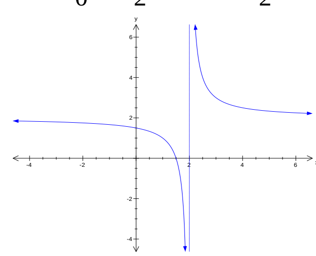
x-intercept: $y = \dfrac{1}{x-2} + 2 = 0$

$\therefore \dfrac{1}{x-2} = -2$

$\therefore x - 2 = -\dfrac{1}{2}, \ \therefore x = \dfrac{3}{2}$

y-intercept at

$y = \dfrac{1}{0-2} + 2 = \dfrac{3}{2}$



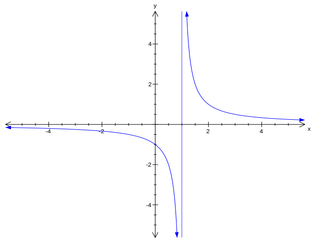
2 $y = f(x) = \frac{1}{x}$

a $y = f(x-1) = \frac{1}{x-1}$

Asymptotes at $x = 1$ and $y = 0$

y-intercept: $y = \frac{1}{0-1} = -1$

No x-intercept because $y = 0$ is an asymptote.



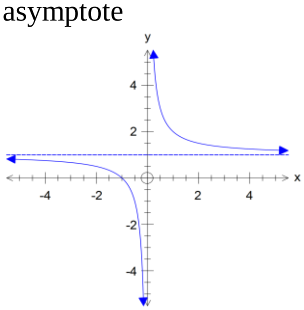
b $y = f(x) + 1 = \frac{1}{x} + 1$

Asymptotes at $x = 0$ and $y = 1$

x intercept: $y = \frac{1}{x} + 1 = 0$

$\therefore \frac{1}{x} = -1, \therefore x = -1$

No y intercept because $y = 0$ is an asymptote

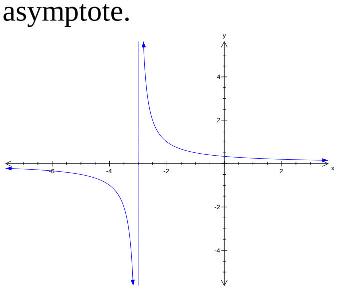


c $y = f(x+3) = \frac{1}{x+3}$

Asymptotes at $x = -3$ and $y = 0$

y-intercept: $y = \frac{1}{0+3} = \frac{1}{3}$

No x-intercept because $y = 0$ is an asymptote.



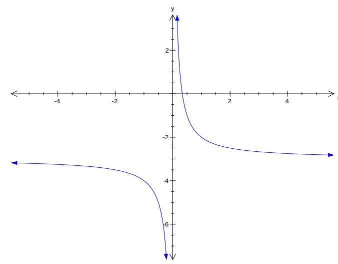
d $y = f(x) - 3 = \frac{1}{x} - 3$

Asymptotes at $x = 0$ and $y = -3$

x-intercept: $y = \frac{1}{x} - 3 = 0$

$\therefore \frac{1}{x} = 3, \therefore x = \frac{1}{3}$

No y-intercept because $x = 0$ is an asymptote.

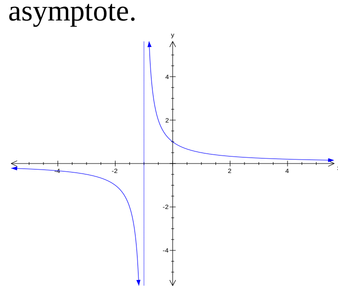


e $y = f(x+1) = \frac{1}{x+1}$

Asymptotes at $x = -1$ and $y = 0$

y-intercept: $y = \frac{1}{0+1} = 1$

No x-intercept because $y = 0$ is an asymptote.



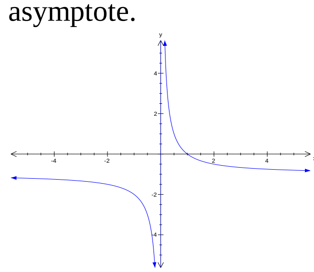
f $y = f(x) - 1 = \frac{1}{x} - 1$

Asymptotes at $x = 0$ and $y = -1$

x-intercept: $y = \frac{1}{x} - 1 = 0$

$\therefore \frac{1}{x} = 1, \therefore x = 1$

No y-intercept because $x = 0$ is an asymptote.



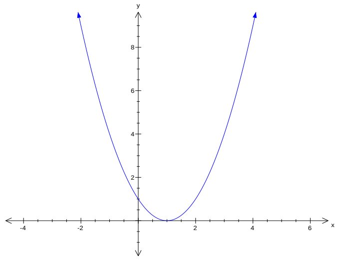
3 $y = f(x) = x^2$

a $y = f(x-1) = (x-1)^2$

x-intercept: $(x-1)^2 = 0, \therefore x = 1$

y-intercept: $f(0-1) = 1$

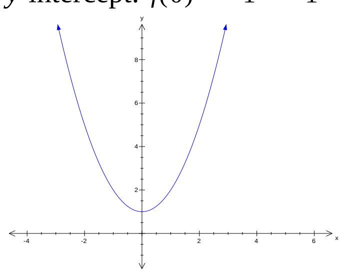
$y = (x-1)^2 = 0, \therefore x = 1$



b $y = f(x) + 1 = x^2 + 1$

No x-intercept because $f(x + 1) > 0$ for all real x.

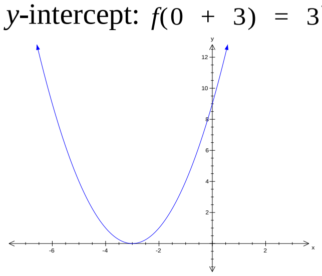
y-intercept: $f(0) + 1 = 1$



c $y = f(x+3) = (x+3)^2$

x-intercept: $(x+3)^2 = 0, \therefore x = -3$

y-intercept: $f(0+3) = 3^2 = 9$



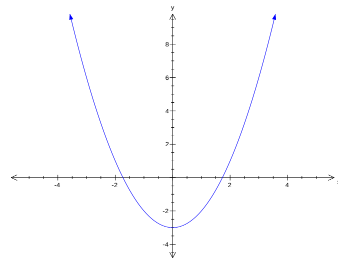
d $y = f(x) - 3 = x^2 - 3$

x-intercepts:

$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$

$\therefore x^2 = 3, \therefore x = \pm\sqrt{3}$

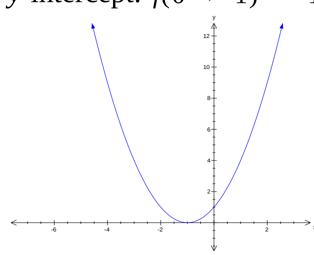
y-intercept: $f(0) - 3 = -3$



e $y = f(x+1) = (x+1)^2$

x-intercept: $(x+1)^2 = 0, \therefore x = -1$

y-intercept: $f(0+1) = 1^2 = 1$



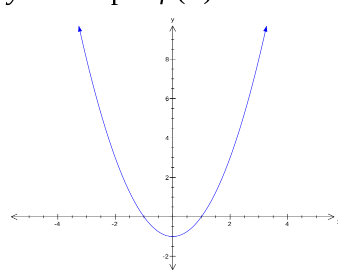
f $y = f(x) - 1 = x^2 - 1$

x-intercepts:

$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$

$\therefore x^2 = 1, \therefore x = \pm 1$

y-intercept: $f(0) - 1 = -1$

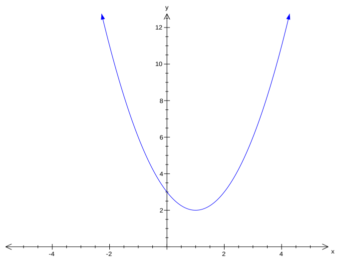


4 $y = f(x) = x^2$

a $y = f(x - 1) + 2 = (x - 1)^2 + 2$

No x intercepts because $f(x - 1) + 2 > 0$ for all real x.

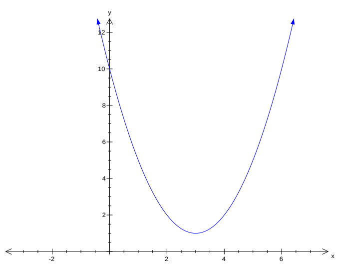
y-intercept: $f(0-1)+1 = (-1)^2 + 2 = 3$



b $y = f(x - 3) + 1 = (x - 3)^2 + 1$

No x-intercepts because $f(x - 3) + 1 > 0$ for all real x.

y-intercept: $f(0-3)+1 = (-3)^2 + 1 = 10$



c $y = f(x + 3) - 5 = (x + 3)^2 - 5$

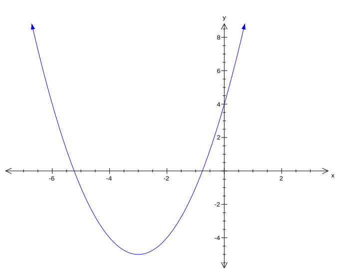
x-intercepts: $y = f(x + 3) - 5 = 0$

$\therefore (x + 3)^2 - 5 = 0$

$\therefore (x + 3)^2 = 5$

$\therefore x + 3 = \pm\sqrt{5}$, $\therefore x = -3 \pm \sqrt{5}$

y-intercept: $f(0 + 3) - 5 = 9 - 5 = 4$



d $y = f(x + 1) - 3 = (x + 1)^2 - 3$

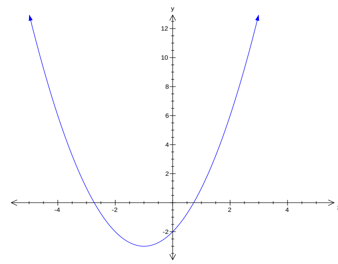
x-intercepts: $y = f(x + 1) - 3 = 0$

$\therefore (x + 1)^2 - 3 = 0$

$\therefore (x + 1)^2 = 3$

$\therefore x + 1 = \pm\sqrt{3}$, $\therefore x = -1 \pm \sqrt{3}$

y-intercept: $f(0 + 1) - 3 = 1 - 3 = -2$



e $y + 2 = f(x + 1)$, $\therefore y = f(x + 1) - 2$

$y = f(x + 1) - 2 = (x + 1)^2 - 2$

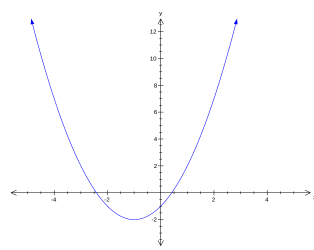
x-intercepts: $y = f(x + 1) - 2 = 0$

$\therefore (x + 1)^2 - 2 = 0$

$\therefore (x + 1)^2 = 2$

$\therefore x + 1 = \pm\sqrt{2}$, $\therefore x = -1 \pm \sqrt{2}$

y-intercept: $f(0 + 1) - 2 = 1 - 2 = -1$



f $y = f(x - 5) - 1 = (x - 5)^2 - 1$

x-intercepts: $y = f(x - 5) - 1 = 0$

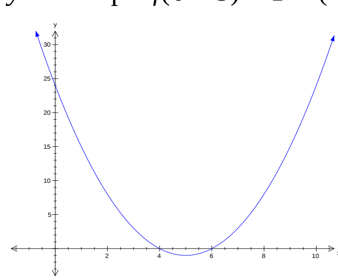
$\therefore (x - 5)^2 - 1 = 0$

$\therefore (x - 5)^2 = 1$

$\therefore x - 5 = \pm 1$

$\therefore x = 5 \pm 1 = 4 ; 6$

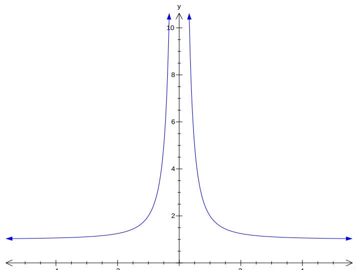
y-intercept: $f(0 - 5) - 1 = (-5)^2 - 1 = 24$



5

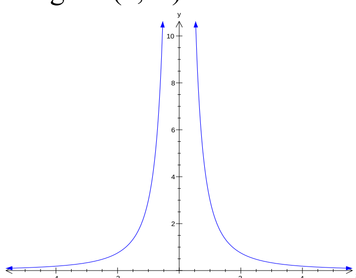
a $y = \frac{1}{x^2} + 1$

Asymptotes at $x = 0$ and $y = 1$
No x-intercept since $y > 1$ for all real x.
No y-intercept since $x = 0$ is an asymptote.
Range is $(1, \infty)$.



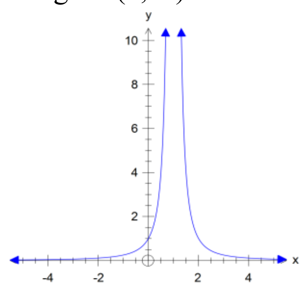
b $y = \frac{3}{x^2}$

Asymptotes at $x = 0$ and $y = 0$
No x-intercept since $y > 0$ for all real x.
No y-intercept since $x = 0$ is an asymptote.
Range is $(0, \infty)$.



c $y = \frac{1}{(x-1)^2}$

Asymptotes at $x = 1$ and $y = 0$
No x-intercept since $y > 0$ for all real x.
y-intercept at $y = \frac{1}{(0-1)^2} = 1$
Range is $(0, \infty)$.



d $y = \frac{1}{x^2} - 4$

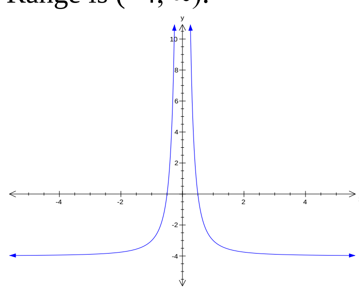
Asymptotes at $x = 0$ and $y = -4$
x-intercept where $y = \frac{1}{x^2} - 4 = 0$

$\therefore \frac{1}{x^2} = 4$

$\therefore x^2 = \frac{1}{4}$

$\therefore x = \pm\frac{1}{2}$

No y-intercept since $x = 0$ is an asymptote.
Range is $(-4, \infty)$.



Exercise 6I Solutions

1

a $y = x^2$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \left(\frac{x}{0.5}\right)^2 = 4x^2$

ii A dilation of factor 5 from the y-axis

$\therefore y = \left(\frac{x}{5}\right)^2 = \frac{x^2}{25}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}(x)^2 = \frac{2x^2}{3}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4(x)^2 = 4x^2$

v A reflection in the x-axis

$\therefore y = -(x)^2 = -x^2$

vi A reflection in the y-axis

$\therefore y = (-x)^2 = x^2$

b $y = \frac{1}{x^2}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \left(\frac{0.5}{x}\right)^2 = \frac{1}{4x^2}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \left(\frac{5}{x}\right)^2 = \frac{25}{x^2}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\left(\frac{1}{x^2}\right) = \frac{2}{3x^2}$

iv A dilation of factor 4 from the x-axis

$\therefore y = \frac{4}{x^2}$

v A reflection in the x-axis

$\therefore y = -\frac{1}{x^2}$

vi A reflection in the y-axis

$\therefore y = \left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}$

c $y = \frac{1}{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \frac{0.5}{x} = \frac{1}{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \frac{5}{x}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\left(\frac{1}{x}\right) = \frac{2}{3x}$

iv A dilation of factor 4 from the x-axis

$\therefore y = \frac{4}{x}$

v A reflection in the x-axis

$\therefore y = -\frac{1}{x}$

vi A reflection in the y-axis

$\therefore y = \frac{1}{-x} = -\frac{1}{x}$

d $y = \sqrt{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \sqrt{\frac{x}{0.5}} = \sqrt{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \sqrt{\frac{x}{5}}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\sqrt{x}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4\sqrt{x}$

v A reflection in the x-axis

$\therefore y = -\sqrt{x}$

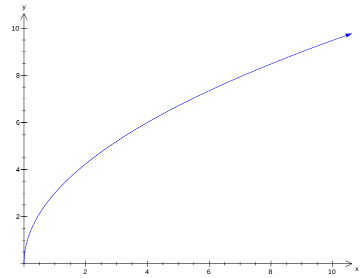
vi A reflection in the y-axis

$\therefore y = \sqrt{-x}\,;\ x \le 0$

2

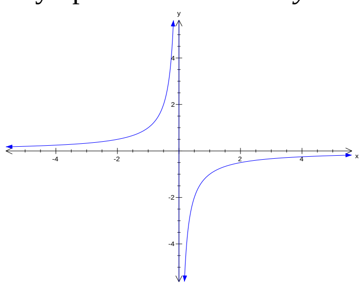
a $y = 3\sqrt{x}$

Starting point at (0,0)



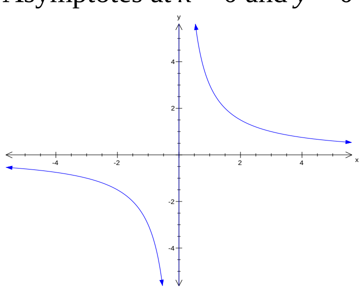
b $y = -\dfrac{1}{x}$

Asymptotes at $x = 0$ and $y = 0$



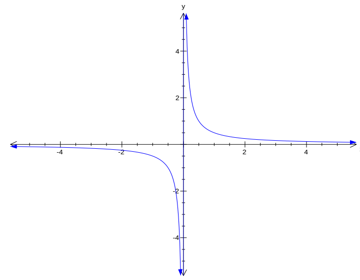
c $y = \dfrac{3}{x}$

Asymptotes at $x = 0$ and $y = 0$



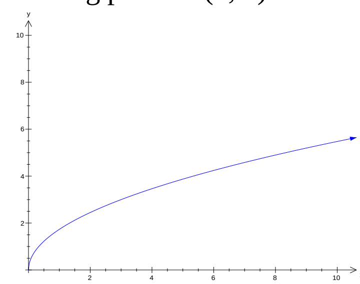
d $y = \dfrac{1}{2x}$

Asymptotes at $x = 0$ and $y = 0$



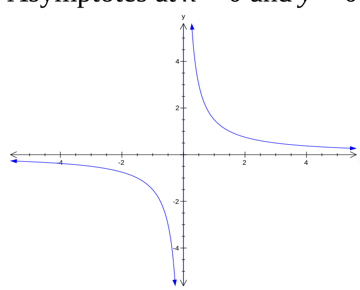
e $y = \sqrt{3x}$

Starting point at (0, 0)



f $y = \dfrac{3}{2x}$

Asymptotes at $x = 0$ and $y = 0$



Exercise 6J Solutions

1

a Translation of 2 units in the positive direction of the x-axis:

$y = \sqrt{x}$ becomes $y = \sqrt{x - 2}$

followed by a dilation of factor 3 from the x-axis: $y = 3\sqrt{x - 2}$

b Translation of 3 units in the negative direction of the x-axis:

$y = \sqrt{x}$ becomes $y = \sqrt{x + 3}$

followed by a reflection in the x-axis:

$y = -\sqrt{x + 3}$

c Reflection in the x-axis:

$y = \sqrt{x}$ becomes $y = -\sqrt{x}$

followed by a dilation of factor 3 from the x-axis: $y = -3\sqrt{x}$

d Reflection in the x-axis: $y = -\sqrt{x}$

followed by a dilation of factor 2 from the y-axis:

$$y = -\sqrt{\frac{x}{2}}$$

e Dilation of factor 2 from the x-axis:

$y = 2\sqrt{x}$

followed by a translation of 2 units in the positive direction of the x-axis:

$y = 2\sqrt{x - 2}$

and 3 units in the negative direction of the y-axis: $y = 2\sqrt{x - 2} - 3$

f Dilation of factor 2 from the y-axis:

$$y = \sqrt{\frac{x}{2}}$$

followed by a translation of 2 units in the negative direction of the x-axis:

$$y = \sqrt{\frac{x + 2}{2}}$$

and 3 units in the negative direction of the y-axis:

$$y = \sqrt{\frac{x + 2}{2}} - 3$$

2 $y = \frac{1}{x}$

a Translation of 2 units in the positive direction of the x-axis:

$y = \frac{1}{x}$ becomes $y = \frac{1}{x - 2}$

followed by a dilation of factor 3 from the x-axis: $y = \frac{3}{x - 2}$

b Translation of 3 units in the negative direction of the x-axis: $y = \frac{1}{x + 3}$

followed by a reflection in the x-axis:

$y = -\frac{1}{x + 3}$

c Reflection in the x-axis: $y = -\frac{1}{x}$

followed by a dilation of factor 3 from the x-axis: $y = -\frac{3}{x}$

d Reflection in the x-axis: $y = -\frac{1}{x}$

followed by a dilation of factor 2 from the y-axis: $y = -\frac{2}{x}$

e Dilation of factor 2 from the x-axis:

$y = \frac{2}{x}$

followed by a translation of 2 units in the positive direction of the x-axis:

$y = \frac{2}{x - 2}$

and 3 units in the negative direction of the y-axis: $y = \frac{2}{x - 2} - 3$

f Dilation of factor 2 from the y-axis:

$y = \frac{2}{x}$

followed by a translation of 2 units in the negative direction of the x-axis:

$y = \frac{2}{x + 2}$

and 3 units in the negative direction of the y-axis: $y = \frac{2}{x + 2} - 3$

3

a **i** $y = 2(x - 1)^2 + 3$ from $y = x^2$

Dilation of factor 2 from the x-axis:

$y = 2x^2$

translation of 1 unit in the positive direction of the x-axis:

$y = 2(x - 1)^2$

and 3 units in the positive direction of the y-axis:

$y = 2(x - 1)^2 + 3$

ii $y = -(x + 1)^2 + 2$

Reflection in the x-axis:

$y = -x^2$

translation of 1 unit in the negative direction of the x-axis:

$y = -(x + 1)^2$

and 2 units in the positive direction of the y-axis:

$y = -(x + 1)^2 + 2$

iii $y = (2x + 1)^2 - 2$

Dilation of factor $\frac{1}{2}$ from the y-axis:

$y = (2x)^2$

translation of $\frac{1}{2}$ unit in the negative direction of the x-axis:

$y = (2x + 1)^2$

and 2 units in the negative direction of the y-axis:

$y = (2x + 1)^2 - 2$

b **i** $y = \dfrac{2}{x + 3}$ from $y = \dfrac{1}{x}$

Dilation of factor 2 from the x-axis:

$y = \dfrac{2}{x}$

translation of 3 units in the negative direction of the x-axis:

$y = \dfrac{2}{x + 3}$

ii $y = \dfrac{1}{x + 3} + 2$

Translation of 3 units in the negative direction of the x-axis:

$y = \dfrac{1}{x + 3}$

and 2 units in the positive direction of the y-axis:

$y = \dfrac{1}{x + 3} + 2$

iii $y = \dfrac{1}{x - 3} - 2$

Translation of 3 units in the positive direction of the x-axis:

$y = \dfrac{1}{x - 3}$

and 2 units in the negative direction of the y-axis:

$y = \dfrac{1}{x - 3} - 2$

c **i** $y = \sqrt{x + 3} + 2$ from $y = \sqrt{x}$

Translation of 3 units in the negative direction of the x-axis:

$y = \sqrt{x + 3}$

and 2 units in the positive direction of the y-axis:

$y = \sqrt{x + 3} + 2$

ii $y = 2\sqrt{3x}$

Dilation of factor $\frac{1}{3}$ from the y-axis:

$y = \sqrt{3x}$

dilation of factor 2 from the x-axis:

$y = 2\sqrt{3x}$

iii $y = -\sqrt{x} + 2$

Reflection in the x-axis:

$y = -\sqrt{x}$

translation of 2 units in the positive direction of the y-axis:

$y = -\sqrt{x} + 2$

Exercise 6K Solutions

1 **a** **i** $A = (8 + x)y - x^2$

ii $P = y + (8 + x) + (y - x) + x + x + 8 = 2x + 2y + 16$

b **i** If $P = 64$, $64 = 2x + 2y + 16$

$\therefore$ $48 = 2(x + y)$

$\therefore$ $24 = x + y$

$\therefore$ $y = 24 - x$

When $y = 24 - x$, $A = (8 + x)(24 - x) - x^2 = 192 + 16x - 2x^2$

ii We know $y = 24 - x$, $\therefore$ $x < 24$

Also $y - x > 0$, i.e. $24 - 2x > 0$ $\therefore$ $x < 12$

The allowable values for x are $\{x: 0 < x < 12\}$.

iii Turning point is at $x = \dfrac{-b}{2a}$

and $a = -2$, $b = 16$ $\therefore$ $x = \dfrac{-16}{-4}$

$= 4$

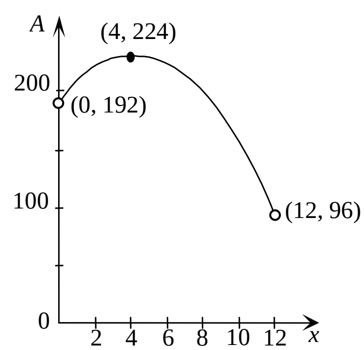
When $x = 4$, $A = 192 + 16(4) - 2(4)^2$

$= 192 + 64 - 32 = 224$

When $x = 0$, $A = 192$

When $x = 12$, $A = 192 + 16(12) - 2(12)^2$

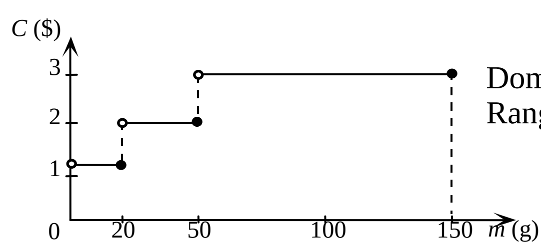
$= 192 + 192 - 288 = 96$



iv The maximum area occurs at the turning point and is 224 cm^2.

2 **a**
$$C = \begin{cases} 1.2 & 0 < m \le 20 \\ 2 & 20 < m \le 50 \\ 3 & 50 < m \le 150 \end{cases}$$

b

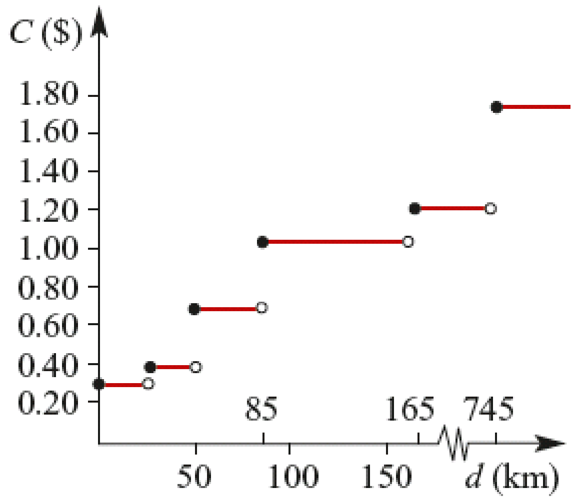


Domain = (0, 150]

Range = {1.20, 2.00, 3.00}

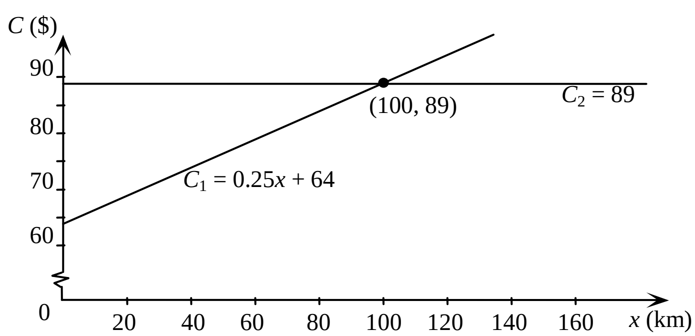
3 **a**
$$C = \begin{cases} 0.3 & 0 \le d < 25 \\ 0.4 & 25 \le d < 50 \\ 0.7 & 50 \le d < 85 \\ 1.05 & 85 \le d < 165 \\ 1.22 & 165 \le d < 745 \\ 1.77 & d \ge 745 \end{cases}$$

b



4 **a** $C_1 = 0.25x + 64$
$C_2 = 89$

b
$$\begin{aligned} 0.25x + 64 &= 89 \\ \text{implies} \quad 0.25x &= 25 \\ \therefore \quad x &= 100 \end{aligned}$$



c Method 2 is cheaper than Method 1 if more than 100 km per day is travelled.

Chapter Review: Multiple-choice Solutions

1 $f(x) = 10x^2 + 2$
$\therefore f(2a) = 10(2a)^2 + 2$
$= 40a^2 + 2$ **B**

2 Maximal domain of $f(x) = \sqrt{3x + 5}$ is
$[-\frac{5}{3}, \infty)$ **E**

3 Range of $x^2 + y^2 > 9$ is *all* numbers outside the circle $x^2 + y^2 = 9$
Hence range is *R*. **B**

4 For $f(x) = 7x - 6$, $f^{-1}(x)$:
$x = 7y - 6$
$x + 6 = 7y$
$\therefore y = \frac{x + 6}{7}$ **C**

5 For f: $(a, b] \rightarrow R$, $f(x) = 3 - x$
Max. value of range $> 3 - a$
Min. value of range $= 3 - b$ **E**

6 **A** $f(x) = 9 - x^2$ is $1 \rightarrow 1$ over $x \geq 0$
B $f(x) = \sqrt{9 - x^2}$ is many $\rightarrow 1$ for implied domain $[-3,3]$.
C $f(x) = 1 - 9x$ is a line and $1 \rightarrow 1$
D $f(x) = \sqrt{x}$ is $1 \rightarrow 1$
E $f(x) = \frac{9}{x}$ is $1 \rightarrow 1$ if domain is $R/\{0\}$

B is correct. **B**

7 $y = \frac{2}{x} + 3$ is reflected in the x-axis:
$y = -\frac{2}{x} - 3$
and then in the y-axis:
$y = -\frac{2}{-x} - 3 = \frac{2}{x} - 3$ **D**

8 $y = x^2$ to $y = -(2x - 6)^2 + 4$:
reflection in the x-axis: $y = -x^2$
dilation of $\frac{1}{2}$ from the y-axis:
$y = -(2x)^2$
translation of 3 units in the positive direction of the x-axis:
$y = -(2(x - 3))^2$
and 4 units in the positive direction of the y-axis:
$y = -(2x - 6)^2 + 4$ **E**

9 For f: $[-1, 5) \rightarrow R$, $f(x) = x^2$
Min. value at $(0, 0)$;
$f(-1) = 1$; $f(5) = 25$:
the range is $[0, 25)$. **C**

10 **A** $y = x^2 - x$ is a many $\rightarrow$ 1 function
B $y = \sqrt{4 - x^2}$ is a many $\rightarrow$ 1 function
C $y = 3$, $x > 0$ is a many $\rightarrow$ 1 function
D $x = 3$ is a $1 \rightarrow$ many relation
E $y = 3x$ is a $1 \rightarrow 1$ function

D is correct. **D**

Chapter Review: Short-answer Solutions (technology-free)

1

a $f(3) = 2 - 6(3) = -16$

b $f(-4) = 2 - 6(-4) = 26$

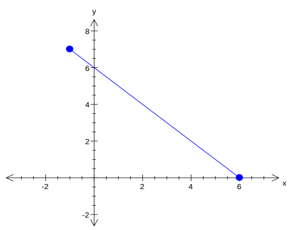
c $f(x) = 2 - 6x = 6$

$\therefore \quad -6x = 4$

$\therefore \quad x = -\dfrac{2}{3}$

2 $f: [-1, 6] \rightarrow R, f(x) = 6 - x$

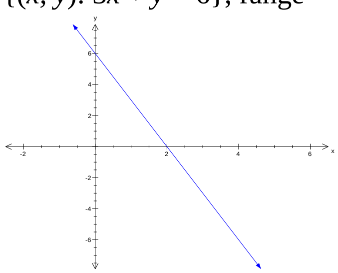
a



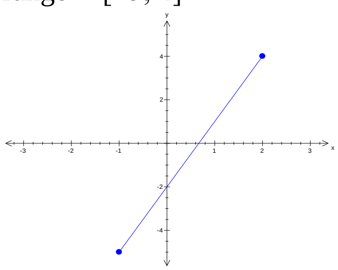
b Range of $f = [0, 7]$

3

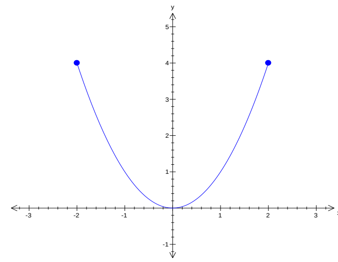
a $\{(x, y): 3x + y = 6\}$; range = R



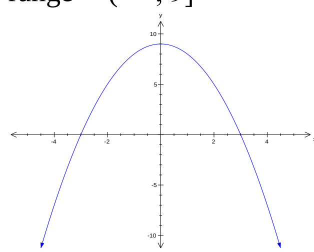
b $\{(x, y): y = 3x - 2;\ x \in [-1, 2]\}$;
range = $[-5, 4]$



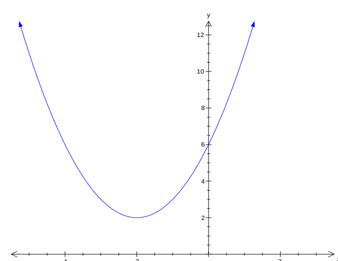
c $\{(x, y): y = x^2;\ x \in [-2, 2]\}$;
range = $[0, 4]$



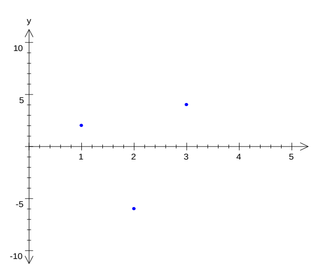
d $\{(x, y): y = 9 - x^2\}$;
range = $(-\infty, 9]$



e $\{(x, y): y = x^2 + 4x + 6\}$;
range = $[2, \infty)$

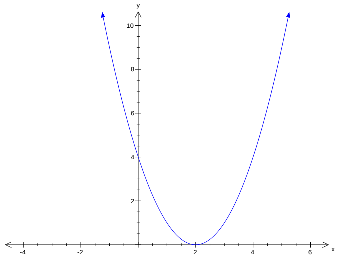


f $\{(1, 2)\ (3, 4)\ (2, -6)\}$;
range = $\{-6, 2, 4\}$



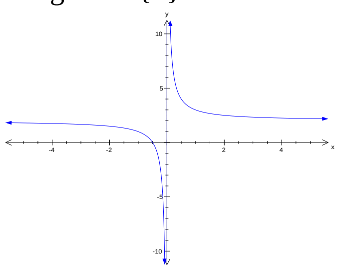
g $f: R \rightarrow R,\ f(x) = (x-2)^2$

Range = $[0, \infty)$



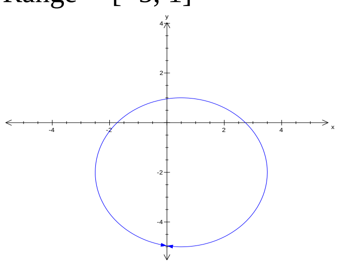
h $f: R\backslash\{0\} \rightarrow R,\ f(x) = \dfrac{1}{x} + 2$

Range = $R\backslash\{2\}$



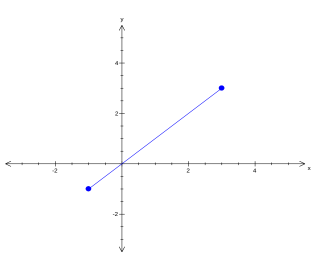
i $\left(x - \dfrac{1}{2}\right)^2 + (y+2)^2 = 9$

Range = $[-5, 1]$



j $f: [-1, 3] \rightarrow R,\ f(x) = x$

Range = $[-1, 3]$



4

a $f(x) = \dfrac{a}{x} + b$

$f(1) = \dfrac{3}{2},\ \therefore f(1) = a + b = \dfrac{3}{2}$

$\therefore b = \dfrac{3}{2} - a$

$f(2) = 9,\ \therefore f(2) = \dfrac{a}{2} + b = 9$

$\therefore \dfrac{a}{2} + \left(\dfrac{3}{2} - a\right) = 9$

$\therefore \dfrac{3}{2} - \dfrac{a}{2} = 9$

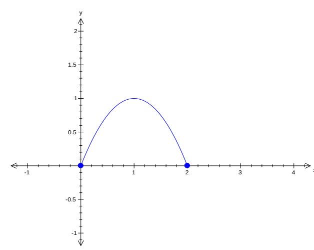
$\therefore 3 - a = 18$

$\therefore a = -15\ ;\ b = \dfrac{33}{2}$

b Implied domain of f is $R\backslash\{0\}$.

5 $f: [0, 2] \rightarrow R,\ f(x) = 2x - x^2$

a



b Range = $[0, 1]$

6 $f(x) = ax + b$

$$\begin{array}{ll} f(5) = 10, & \therefore 5a + b = 10 \\ f(1) = -2, & \therefore \ \ a + b = -2 \\ \hline & \therefore 4a \qquad = 12 \end{array}$$

$a = 3,\ b = -5$

7 $f(x) = ax^2 + bx + c$

$f(0) = 0,\ \therefore c = 0$

$$\begin{array}{ll} f(4) = 0 & \therefore 16a + 4b = 0 \\ & \therefore 4a + b = 0 \\ f(-2) = -6 & \therefore 4a - 2b = -6 \\ \hline & \therefore \qquad 3b = 6 \end{array}$$

$\therefore b = 2;\ 4a = -2$

$a = -\dfrac{1}{2},\ b = 2,\ c = 0$

8

a $y = \dfrac{1}{x-2}$;
implied domain = $R\backslash\{2\}$

b $f(x) = \sqrt{x-2}$;
implied domain = $[2, \infty)$

c $y = \sqrt{25 - x^2}$;
implied domain = $[-5, 5]$ since
$25 - x^2 \geq 0$

d $f(x) = \dfrac{1}{2x-1}$;
implied domain = $R\backslash\{\frac{1}{2}\}$

e $g(x) = \sqrt{100 - x^2}$;
implied domain = $[-10, 10]$

f $h(x) = \sqrt{4 - x}$;
implied domain = $(-\infty, 4]$

9

a $y = x^2 + 2x + 3$ is many $\rightarrow$ 1
(full parabola)

b $f: [2, \infty) \rightarrow R$, $f(x) = (x-2)^2$ is $1 \rightarrow 1$
since we only have the right side of the parabola.

c $f(x) = 3x + 2$ is $1 \rightarrow 1$ (oblique line)

d $f(x) = \sqrt{x-2}$ is $1 \rightarrow 1$
(half-parabola only)

e $f(x) = \dfrac{1}{x-2}$ is $1 \rightarrow 1$
(rectangular hyperbola)

f $f: [-1, \infty) \rightarrow R$, $f(x) = (x+2)^2$ is $1 \rightarrow 1$
since we only have part of the right side of the parabola (vertex at $x = -2$).

g $f: [-3, 5] \rightarrow R$, $f(x) = 3x - 2$ is $1 \rightarrow 1$
(oblique line)

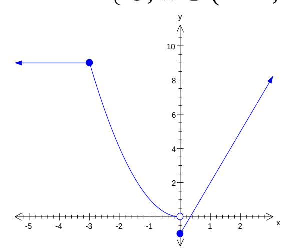
h $f(x) = 7 - x^2$ is many $\rightarrow$ 1
(full parabola)

i $f(x) = \dfrac{1}{(x-2)^2}$ is many $\rightarrow$ 1
(full truncus)

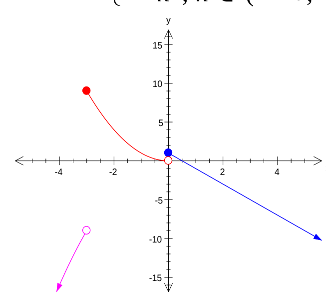
j $h(x) = \dfrac{1}{x-2} + 4$ is $1 \rightarrow 1$
(rectangular hyperbola)

10

a $f(x) = \begin{cases} 3x - 1;\ x \in [0, \infty) \\ x^2;\ x \in [-3, 0) \\ 9;\ x \in (-\infty, -3) \end{cases}$



b $h(x) = \begin{cases} 1 - 2x;\ x \in [0, \infty) \\ x^2;\ x \in [-3, 0) \\ -x^2;\ x \in (-\infty, -3) \end{cases}$



11

a $f: [-1, 5] \to R, f(x) = 3x - 2$

Range of $f = [-5, 13]$

$= \text{domain of inverse}$

$x = 3y - 2$

$\therefore 3y = x + 2$

$\therefore y = \frac{x+2}{3}$

$\therefore f^{-1}(x) = \frac{x+2}{3}$

Domain = $[-5, 13]$

b $f: [-2, \infty) \to R, f(x) = \sqrt{x+2} + 2$

Range of $f = [2, \infty)$

$= \text{domain of inverse}$

$x = \sqrt{y+2} + 2$

$\therefore x - 2 = \sqrt{y+2}$

$\therefore y + 2 = (x-2)^2$

$\therefore f^{-1}(x) = (x-2)^2 - 2$

or $(\text{or } x^2 - 4x + 2)$

Domain = $[2, \infty)$

c $f: [-1, \infty) \to R, f(x) = 3(x+1)^2$

Range of $f = [0, \infty)$

$= \text{domain of inverse}$

$x = 3(y+1)^2$

$\therefore (y+1)^2 = \frac{x}{3}$

$\therefore y + 1 = \sqrt{\frac{x}{3}}$

$\therefore f^{-1}(x) = \sqrt{\frac{x}{3}} - 1$

Domain = $[0, \infty)$

d $f: (-\infty, 1) \to R, f(x) = (x-1)^2$

Range of $f = [0, \infty)$

$= \text{domain of inverse}$

$x = (y-1)^2$

$\therefore -\sqrt{x} = y - 1$

$\therefore f^{-1}(x) = 1 - \sqrt{x}$

Domain = $[0, \infty)$

We need the negative root here because $f(x)$ is the left side of the parabola.

12 $f(x) = \sqrt{x}$

a Translation of 2 in the positive direction of the x-axis:

$y = \sqrt{x-2}$

and 3 in the positive direction of the y-axis:

$y = \sqrt{x-2} + 3$

b Dilation of factor 2 from the x-axis:

$y = 2\sqrt{x}$

c Reflection in the x-axis:

$y = -\sqrt{x}$

d Reflection in the y-axis:

$y = \sqrt{-x}$

e Dilation of factor 3 from the y-axis:

$y = \sqrt{\frac{x}{3}}$

Chapter Review: Extended-response Solutions

1 **a** For the first coach,

$$d = 80t \quad \text{for } 0 \le t \le 4$$

$$d = 320 \quad \text{for } 4 < t \le 4\tfrac{3}{4}$$

$$d = 320 + 80\left(t - 4\tfrac{3}{4}\right) \quad \text{for } 4\tfrac{3}{4} < t \le 7\tfrac{1}{4}$$

$$= 320 + 80t - 380 = 80t - 60$$

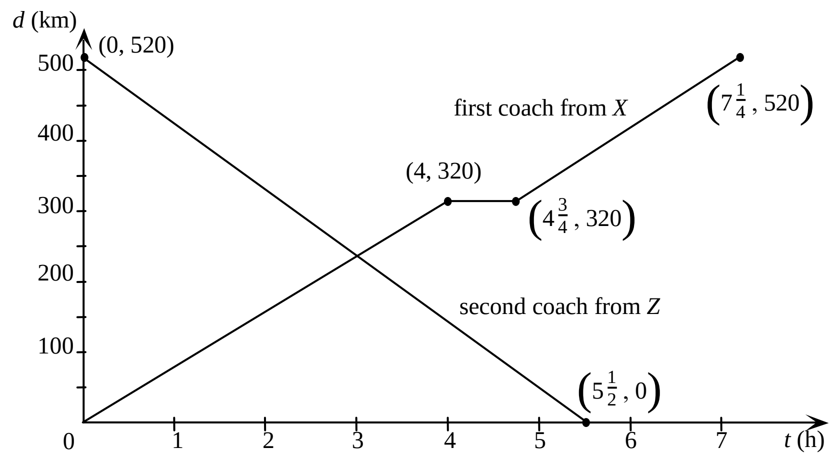
i.e. for the first coach, $d = \begin{cases} 80t & 0 \le t \le 4 \\ 320 & 4 < t \le 4\frac{3}{4} \\ 80t - 60 & 4\frac{3}{4} < t \le 7\frac{1}{4} \end{cases}$ Range: [0, 520]

For the second coach, $v = \dfrac{d}{t}$

$$= \frac{520}{5\frac{1}{2}} = \frac{1040}{11}$$

$\therefore \ d = 520 - \dfrac{1040}{11}t, \quad 0 \le t \le 5\dfrac{1}{2},$ Range: [0, 520]

b The point of intersection of the two graphs yields the time at which the two coaches pass and where this happens.



$$520 - \frac{1040}{11}t = 80t$$

$$\therefore \quad 520 = \frac{1920t}{11}$$

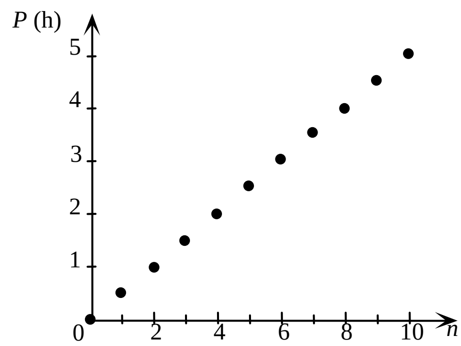
$$\therefore \quad t = \frac{520 \times 11}{1920} = \frac{143}{48}$$

When $t = \dfrac{143}{48}$, $\quad d = 80 \times \dfrac{143}{48}$

$$= \frac{11\,440}{48} = 238\frac{1}{3}$$

i.e. the two coaches pass each other $238\frac{1}{3}$ km from X.

2 **a** $P = \frac{1}{2}n$, $n \le 200$, $n \in Z$



b Domain = $\{n: n \in \text{Z}, 0 \le n \le 200\}$

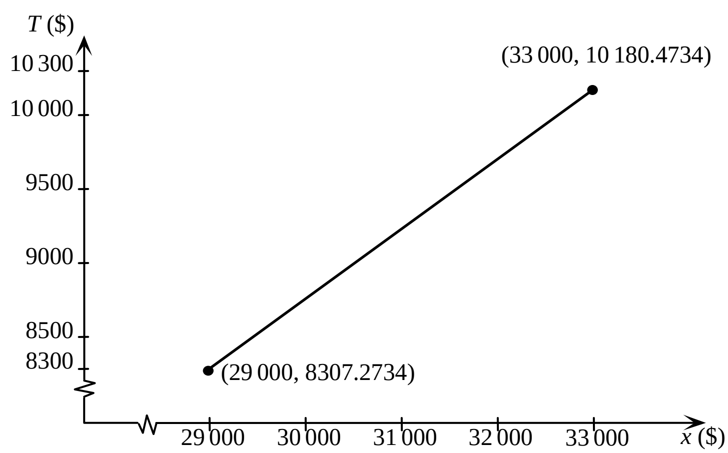
Range = $\{\frac{n}{2} : n \in \text{Z}, 0 \le n \le 200\}$

3 **a** $T = D + 5\,817.73 + 0.4683 \times 1000 + 0.4683(x - 29\,000)$

$= 0.2442(12\,500 - 4223) + 5817.73 + 468.30 + 0.4683x - 13\,580.70$

i.e. $T = 0.4683x - 5273.4266;\ 29\,000 \le x \le 33\,000$

b



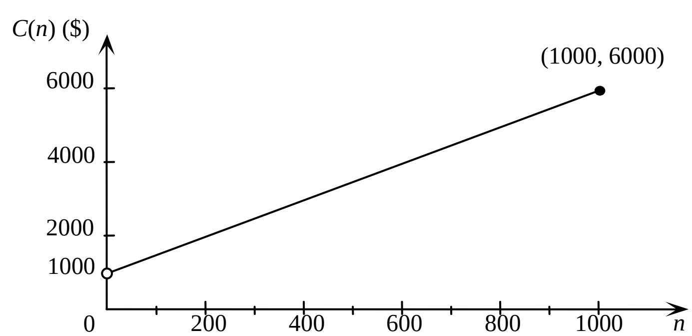
Range = [8 307.2734, 10 180.4734]

c When $x = 30\,000$, $T = 0.4683 \times 30\,000 - 5273.4266 = 8775.5734$

i.e. tax payable on \$30 000 is \$8775.57 (to the nearest cent).

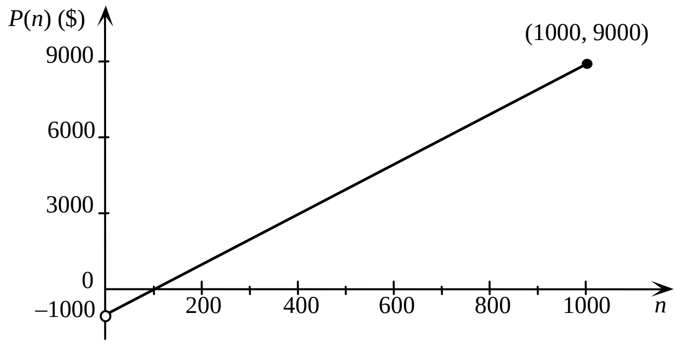
4 **a** **i** $C(n) = 5n + 1000$, $n > 0$

ii



b **i** $P(n) = 15n - C(n)$
$= 15n - (5n + 1000)$
$= 10n - 1000,\ n > 0$

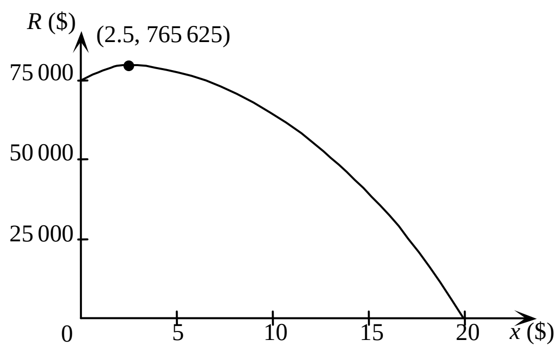
ii



5 $V = 8000 - 0.05 \times 8000 \times n = 8000 - 400n,\ n \geq 0$

6 **a** R = price of ticket × number of tickets sold
$= (15 + x)(50\,000 - 2500x)$
$= 2500(x + 15)(20 - x),\ 0 \leq x \leq 20$

b

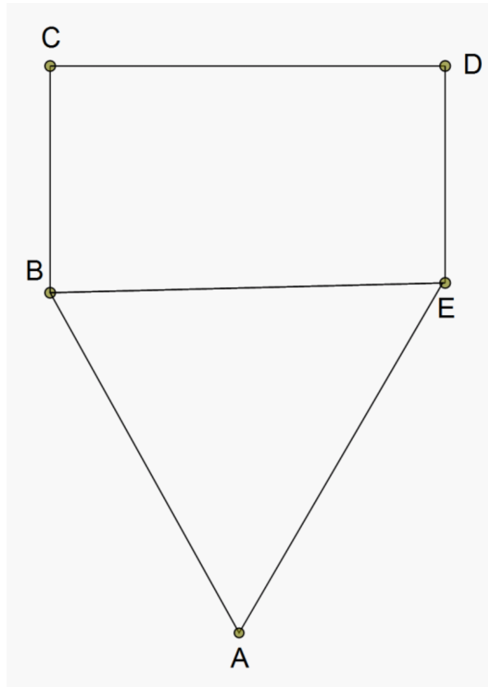


x-intercepts occur when $R = 0$ $\therefore\ x = -15$ or 20 (but $x \geq 0$, so $x = 20$)
R-intercept occurs when $x = 0$ $\therefore\ R = 750\,000$
Turning point occurs when $x = \dfrac{-15 + 20}{2} = 2.5$
When $x = 2.5$, $R = 2500(2.5 + 15)(20 - 2.5) = 765\,625$

c The price which will maximise the revenue is \$17.50 (i.e. when $x = 2.5$).
This assumes that price can be increased by other than dollar amounts.

7 **a** $BE = CD = x$, as $BCDE$ is a rectangle.
$AB = AE = BE = x$,
as ABE is an equilateral triangle.
$DE = BC$, as $BCDE$ is a rectangle.
Perimeter of pentagon
$= 3x + 2BC = a$

$\therefore \quad BC = \frac{1}{2}(a - 3x)$



$$\begin{aligned} A(x) &= \text{area of rectangle + area of triangle} \\ &= \text{length} \times \text{width} + \frac{1}{2}\text{base} \times \text{height} \\ &= x \times \frac{1}{2}(a - 3x) + \frac{1}{2}x \times \sqrt{x^2 - \left(\frac{1}{2}x\right)^2} \\ &= \frac{ax}{2} - \frac{3x^2}{2} + \frac{x}{2}\sqrt{x^2\left(1 - \frac{1}{4}\right)} = \frac{ax}{2} - \frac{3x^2}{2} + \frac{x^2\sqrt{3}}{4} \\ &= \frac{2ax - 6x^2 + x^2\sqrt{3}}{4} \\ &= \frac{x}{4}(2a - (6 - \sqrt{3})x) \end{aligned}$$

b From geometry: $x>0$
Also BC > 0
So a-3x>0
Giving $x < \frac{a}{3}$
Therefore allowable values for x are $\{x: 0 < x < \frac{a}{3}\}$.

c Maximum area occurs when $x = \frac{0 + \frac{2a}{6-\sqrt{3}}}{2} = \frac{a}{6-\sqrt{3}}$

When $x = \frac{a}{6-\sqrt{3}}$, $\quad A(x) = \frac{a}{4(6-\sqrt{3})}\left(2a - \frac{(6-\sqrt{3})a}{6-\sqrt{3}}\right)$

$= \frac{a^2}{4(6-\sqrt{3})}$

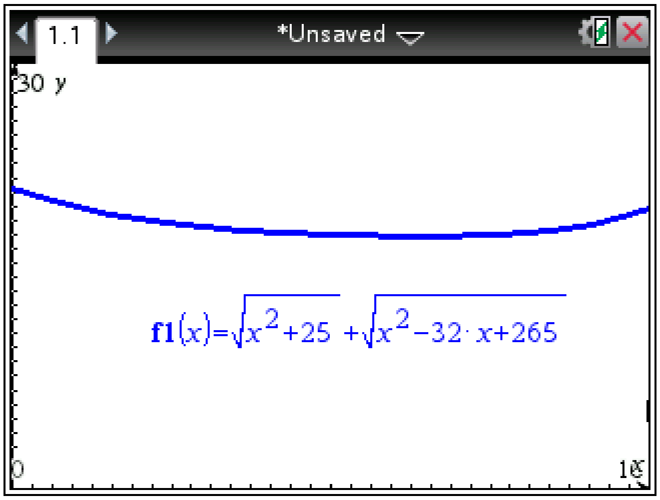
i.e. the maximum area is $\frac{a^2}{4(6-\sqrt{3})}$ cm^2.

8 **a** **i** $d(x) = AP + PD$

$$= \sqrt{AB^2 + BP^2} + \sqrt{PC^2 + CD^2}$$
$$= \sqrt{5^2 + x^2} + \sqrt{(16 - x)^2 + 3^2}$$
$$= \sqrt{x^2 + 25} + \sqrt{x^2 - 32x + 265}$$

ii $0 \le BO \le BC \quad \therefore\ 0 \le x \le 16$

b **i**



ii On a CAS calculator, sketch the graphs of **_f_1=√(_x_^2+25)+√(_x_^2–32_x_+265)** and **_f_2=20**. The point of intersection is (1.539 579 7, 20). Therefore $d(x) = 20$ when $x \approx 1.54$.
Alternatively, enter **solve(√(_x_^2+25)+√(_x_^2–32_x_+265)=20,_x_)** to find

$$x = \frac{80 \pm 25\sqrt{7}}{9} \approx 1.54,\ 16.24$$

However, $0 \le x \le 16$, so just one answer of 1.54 .

iii Repeat **b ii** using **_f2_=19**. The points of intersection are (3.396 850 3, 19) and (15.041 245, 19). Therefore $d(x) = 19$ when $x \approx 3.40$ or $x \approx 15.04$.
Alternatively, enter **solve(√(_x_^2+25)+√(_x_^2–32_x_+265)=19,_x_)** to find

$$x = \frac{1936 \pm 19\sqrt{4141}}{210} \approx 15.04, 3.40 \quad .$$

c **i** with **_f_1=√(_x_^2+25)+√(_x_^2–32_x_+265)**
TI: Press **Menu→6:Analyze Graph→2:Minimum**
CP: Tap **Analysis→G-Solve→Min**
to yield (9.999 999 8, 17.888 544). Therefore the minimum value of $d(x)$ is 17.89 when $x \approx 10.00$.

ii Range = [17.89, 21.28]. Exact answer is $[8\sqrt{5}, 5 + \sqrt{265}]$.

9 **a** On a CAS calculator, sketch the graphs of ***f*1=(*x*+1)(6–*x*)** and ***f*2=2*x***.
Points of intersection are (–1.372 281, –2.744 563) and (4.372 281 3, 8.744 562 6). The coordinates of *A* and *B* are (4.37, 8.74) and (–1.37, –2.74) respectively, correct to 2 decimal places.

Or consider $(x+1)(6-x) = 2x$

$\therefore \quad -x^2 + 5x + 6 = 2x$

$\therefore \quad x^2 - 3x - 6 = 0$

$\therefore \quad x^2 - 3x + \frac{9}{4} - \frac{33}{4} = 0$

$\therefore \quad \left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{33}}{2}\right)^2 = 0$

$\therefore \quad \left(x - \frac{3}{2} + \frac{\sqrt{33}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{33}}{2}\right) = 0$

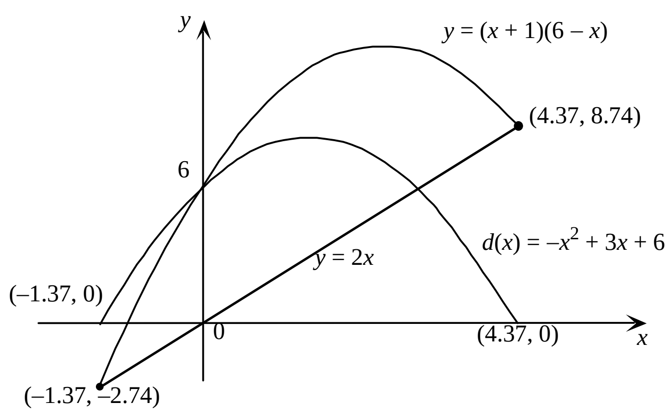
$\therefore \quad x = \frac{3 \pm \sqrt{33}}{2}$ and $y = 3 \pm \sqrt{33}$

yielding $A\left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right)$, $B\left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

b **i**

$$\begin{aligned} d(x) &= (x+1)(6-x) - 2x \\ &= 6x + 6 - x^2 - x - 2x \\ &= -x^2 + 3x + 6 \end{aligned}$$

ii



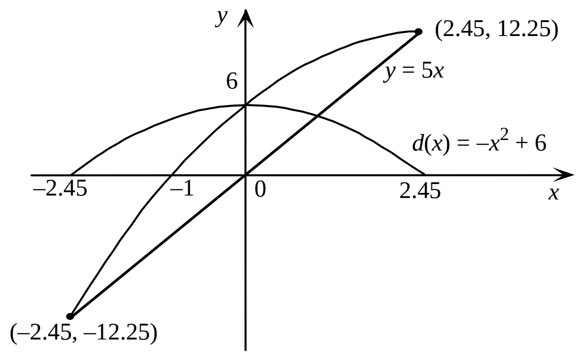
c **i** with ***f*1=–*x*^2+3*x*+6**,
TI: Press **Menu→6:Analyze Graph→3:Maximum**
CP: Tap **Analysis→G-Solve→Max**
to yield (1.500 001 5, 8.25).
Therefore the maximum value of $d(x)$ is 8.25.

ii Range = [0, 8.25]

d $d(x) = (x + 1)(6 - x) - 5x$

$= 6x + 6 - x^2 - x - 5x$

$= -x^2 + 6$



The maximum value
of $d(x)$ is 6
and the range is [0, 6].