

# 6

## The Circular Functions and Their Graphs

In August 2003, the planet Mars passed closer to Earth than it had in almost 60,000 years. Like Earth, Mars rotates on its axis and thus has days and nights. The photos here were taken by the Hubble telescope and show two nearly opposite sides of Mars. (Source: [www.hubblesite.org](http://www.hubblesite.org)) In Exercise 92 of Section 6.2, we examine the length of a Martian day.

Phenomena such as rotation of a planet on its axis, high and low tides, and changing of the seasons of the year are modeled by *periodic functions*. In this chapter, we see how the trigonometric functions of the previous chapter, introduced there in the context of ratios of the sides of a right triangle, can also be viewed from the perspective of motion around a circle.

### 6.1 Radian Measure

### 6.2 The Unit Circle and Circular Functions

### 6.3 Graphs of the Sine and Cosine Functions

### 6.4 Graphs of the Other Circular Functions

### Summary Exercises on Graphing Circular Functions

### 6.5 Harmonic Motion

## 6.1 Radian Measure

**Radian Measure** ■ **Converting Between Degrees and Radians** ■ **Arc Length of a Circle** ■ **Area of a Sector of a Circle**

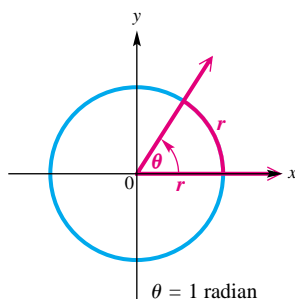


Figure 1

**TEACHING TIP** Students may resist the idea of using radian measure. Explain that radians have wide applications in angular motion problems (seen in Section 6.2), as well as engineering and science.

**Radian Measure** In most applications of trigonometry, angles are measured in degrees. In more advanced work in mathematics, *radian measure* of angles is preferred. Radian measure allows us to treat the trigonometric functions as functions with domains of *real numbers*, rather than angles.

Figure 1 shows an angle  $\theta$  in standard position along with a circle of radius  $r$ . The vertex of  $\theta$  is at the center of the circle. Because angle  $\theta$  intercepts an arc on the circle equal in length to the radius of the circle, we say that angle  $\theta$  has a measure of 1 radian.

### Radian

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of **1 radian**.

It follows that an angle of measure 2 radians intercepts an arc equal in length to twice the radius of the circle, an angle of measure  $\frac{1}{2}$  radian intercepts an arc equal in length to half the radius of the circle, and so on. In general, if  $\theta$  is a central angle of a circle of radius  $r$  and  $\theta$  intercepts an arc of length  $s$ , then the radian measure of  $\theta$  is  $\frac{s}{r}$ .

**Converting Between Degrees and Radians** The circumference of a circle—the distance around the circle—is given by  $C = 2\pi r$ , where  $r$  is the radius of the circle. The formula  $C = 2\pi r$  shows that the radius can be laid off  $2\pi$  times around a circle. Therefore, an angle of  $360^\circ$ , which corresponds to a complete circle, intercepts an arc equal in length to  $2\pi$  times the radius of the circle. Thus, an angle of  $360^\circ$  has a measure of  $2\pi$  radians:

$$360^\circ = 2\pi \text{ radians.}$$

An angle of  $180^\circ$  is half the size of an angle of  $360^\circ$ , so an angle of  $180^\circ$  has half the radian measure of an angle of  $360^\circ$ .

$$180^\circ = \frac{1}{2}(2\pi) \text{ radians} = \pi \text{ radians} \quad \text{Degree/radian relationship}$$

We can use the relationship  $180^\circ = \pi$  radians to develop a method for converting between degrees and radians as follows.

$$180^\circ = \pi \text{ radians}$$

$$1^\circ = \frac{\pi}{180} \text{ radian} \quad \text{Divide by 180.} \quad \text{or} \quad 1 \text{ radian} = \frac{180^\circ}{\pi} \quad \text{Divide by } \pi.$$

### Converting Between Degrees and Radians

1. Multiply a degree measure by  $\frac{\pi}{180}$  radian and simplify to convert to radians.
2. Multiply a radian measure by  $\frac{180^\circ}{\pi}$  and simplify to convert to degrees.

**TEACHING TIP** Emphasize the difference between an angle of  $a$  degrees (written  $a^\circ$ ) and an angle of  $a$  radians (written  $a^r$ ,  $a_r$ , or  $a$ ). Caution students that although radian measures are often given exactly in terms of  $\pi$ , they can also be approximated using decimals.

45°	.7853981634
$\pi/4$	.7853981634
249.8°	4.359832471

Some calculators (in radian mode) have the capability to convert directly between decimal degrees and radians. This screen shows the conversions for Example 1. Note that when *exact* values involving  $\pi$  are required, such as  $\frac{\pi}{4}$  in part (a), calculator approximations are not acceptable.

$(9\pi/4)^r$	405
4.25 <sup>r</sup>	243.5070629

This screen shows how a calculator converts the radian measures in Example 2 to degree measures.

### EXAMPLE 1 Converting Degrees to Radians

Convert each degree measure to radians.

(a)  $45^\circ$

(b)  $249.8^\circ$

**Solution**

(a)  $45^\circ = 45 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{\pi}{4} \text{ radian}$  Multiply by  $\frac{\pi}{180}$  radian.

(b)  $249.8^\circ = 249.8 \left( \frac{\pi}{180} \text{ radian} \right) \approx 4.360 \text{ radians}$  Nearest thousandth

Now try Exercises 1 and 13.

### EXAMPLE 2 Converting Radians to Degrees

Convert each radian measure to degrees.

(a)  $\frac{9\pi}{4}$

(b) 4.25 (Give the answer in decimal degrees.)

**Solution**

(a)  $\frac{9\pi}{4} = \frac{9\pi}{4} \left( \frac{180^\circ}{\pi} \right) = 405^\circ$  Multiply by  $\frac{180^\circ}{\pi}$ .

(b)  $4.25 = 4.25 \left( \frac{180^\circ}{\pi} \right) \approx 243.5^\circ$  Use a calculator.

Now try Exercises 21 and 31.

*If no unit of angle measure is specified, then radian measure is understood.*

**CAUTION** Figure 2 shows angles measuring 30 radians and  $30^\circ$ . Be careful not to confuse them.

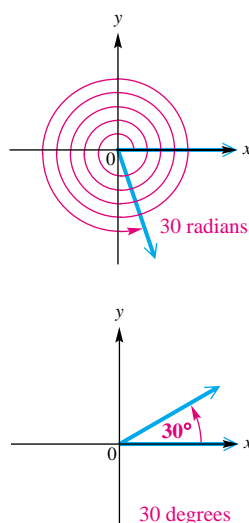


Figure 2

The following table and Figure 3 on the next page give some equivalent angles measured in degrees and radians. Keep in mind that  $180^\circ = \pi$  radians.

Degrees	Radians		Degrees	Radians	
	Exact	Approximate		Exact	Approximate
$0^\circ$	0	0	$90^\circ$	$\frac{\pi}{2}$	1.57
$30^\circ$	$\frac{\pi}{6}$	.52	$180^\circ$	$\pi$	3.14
$45^\circ$	$\frac{\pi}{4}$	.79	$270^\circ$	$\frac{3\pi}{2}$	4.71
$60^\circ$	$\frac{\pi}{3}$	1.05	$360^\circ$	$2\pi$	6.28

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## Looking Ahead to Calculus

In calculus, radian measure is much easier to work with than degree measure. If  $x$  is measured in radians, then the derivative of  $f(x) = \sin x$  is

$$f'(x) = \cos x.$$

However, if  $x$  is measured in degrees, then the derivative of  $f(x) = \sin x$  is

$$f'(x) = \frac{\pi}{180} \cos x.$$

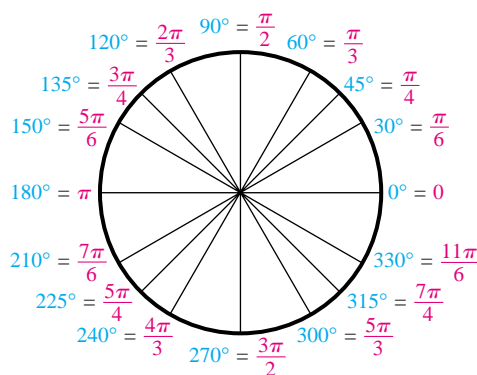


Figure 3

We use radian measure to simplify certain formulas, two of which follow. Each would be more complicated if expressed in degrees.

**Arc Length of a Circle** We use the first formula to find the length of an arc of a circle. This formula is derived from the fact (proven in geometry) that the length of an arc is proportional to the measure of its central angle.

In Figure 4, angle  $QOP$  has measure 1 radian and intercepts an arc of length  $r$  on the circle. Angle  $ROT$  has measure  $\theta$  radians and intercepts an arc of length  $s$  on the circle. Since the lengths of the arcs are proportional to the measures of their central angles,

$$\frac{s}{r} = \frac{\theta}{1}.$$

Multiplying both sides by  $r$  gives the following result.

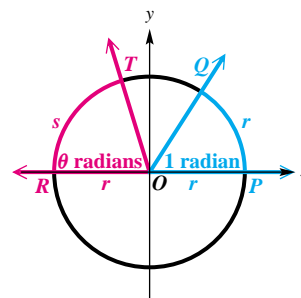


Figure 4

## Arc Length

The length  $s$  of the arc intercepted on a circle of radius  $r$  by a central angle of measure  $\theta$  radians is given by the product of the radius and the radian measure of the angle, or

$$s = r\theta, \quad \theta \text{ in radians.}$$

**TEACHING TIP** To help students appreciate radian measure, show them that in degrees the formula for arc length is

$$s = \frac{\theta}{360} (2\pi r) = \frac{\theta\pi r}{180}.$$

(See Exercise 54.)

**CAUTION** When applying the formula  $s = r\theta$ , the value of  $\theta$  must be expressed in radians.

**EXAMPLE 3** Finding Arc Length Using  $s = r\theta$ 

A circle has radius 18.2 cm. Find the length of the arc intercepted by a central angle having each of the following measures.

(a)  $\frac{3\pi}{8}$  radians

(b)  $144^\circ$

**Solution**

(a) As shown in the figure,  $r = 18.2$  cm and  $\theta = \frac{3\pi}{8}$ .

$$s = r\theta \quad \text{Arc length formula}$$

$$s = 18.2 \left( \frac{3\pi}{8} \right) \text{ cm} \quad \text{Substitute for } r \text{ and } \theta.$$

$$s = \frac{54.6\pi}{8} \text{ cm} \approx 21.4 \text{ cm}$$

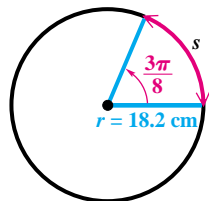
(b) The formula  $s = r\theta$  requires that  $\theta$  be measured in radians. First, convert  $\theta$  to radians by multiplying  $144^\circ$  by  $\frac{\pi}{180}$  radian.

$$144^\circ = 144 \left( \frac{\pi}{180} \right) = \frac{4\pi}{5} \text{ radians} \quad \text{Convert from degrees to radians.}$$

The length  $s$  is given by

$$s = r\theta = 18.2 \left( \frac{4\pi}{5} \right) = \frac{72.8\pi}{5} \approx 45.7 \text{ cm.}$$

Now try Exercises 49 and 51.

**EXAMPLE 4** Using Latitudes to Find the Distance Between Two Cities

Reno, Nevada, is approximately due north of Los Angeles. The latitude of Reno is  $40^\circ$  N, while that of Los Angeles is  $34^\circ$  N. (The N in  $34^\circ$  N means *north* of the equator.) The radius of Earth is 6400 km. Find the north-south distance between the two cities.

**Solution** Latitude gives the measure of a central angle with vertex at Earth's center whose initial side goes through the equator and whose terminal side goes through the given location. As shown in Figure 5, the central angle between Reno and Los Angeles is  $40^\circ - 34^\circ = 6^\circ$ . The distance between the two cities can be found by the formula  $s = r\theta$ , after  $6^\circ$  is first converted to radians.

$$6^\circ = 6 \left( \frac{\pi}{180} \right) = \frac{\pi}{30} \text{ radian}$$

The distance between the two cities is

$$s = r\theta = 6400 \left( \frac{\pi}{30} \right) \approx 670 \text{ km.} \quad \text{Let } r = 6400 \text{ and } \theta = \frac{\pi}{30}.$$

Now try Exercise 55.

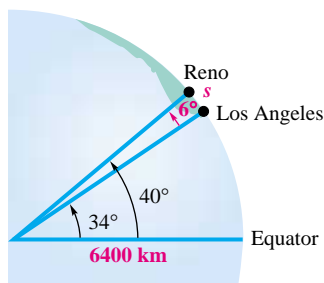


Figure 5

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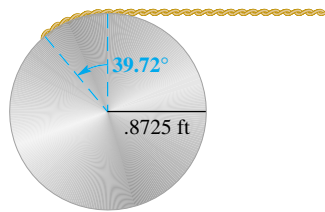


Figure 6

**EXAMPLE 5** Finding a Length Using  $s = r\theta$ 

A rope is being wound around a drum with radius .8725 ft. (See Figure 6.) How much rope will be wound around the drum if the drum is rotated through an angle of  $39.72^\circ$ ?

**Solution** The length of rope wound around the drum is the arc length for a circle of radius .8725 ft and a central angle of  $39.72^\circ$ . Use the formula  $s = r\theta$ , with the angle converted to radian measure. The length of the rope wound around the drum is approximately

$$s = r\theta = .8725 \left[ 39.72 \left( \frac{\pi}{180} \right) \right] \approx .6049 \text{ ft.}$$

Now try Exercise 61(a).

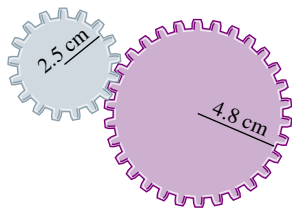


Figure 7

**EXAMPLE 6** Finding an Angle Measure Using  $s = r\theta$ 

Two gears are adjusted so that the smaller gear drives the larger one, as shown in Figure 7. If the smaller gear rotates through  $225^\circ$ , through how many degrees will the larger gear rotate?

**Solution** First find the radian measure of the angle, and then find the arc length on the smaller gear that determines the motion of the larger gear. Since  $225^\circ = \frac{5\pi}{4}$  radians, for the smaller gear,

$$s = r\theta = 2.5 \left( \frac{5\pi}{4} \right) = \frac{12.5\pi}{4} = \frac{25\pi}{8} \text{ cm.}$$

An arc with this length on the larger gear corresponds to an angle measure  $\theta$ , in radians, where

$$s = r\theta$$

$$\frac{25\pi}{8} = 4.8\theta \quad \text{Substitute } \frac{25\pi}{8} \text{ for } s \text{ and } 4.8 \text{ for } r.$$

$$\frac{125\pi}{192} \approx \theta. \quad 4.8 = \frac{48}{10} = \frac{24}{5}; \text{ multiply by } \frac{5}{24} \text{ to solve for } \theta.$$

Converting  $\theta$  back to degrees shows that the larger gear rotates through

$$\frac{125\pi}{192} \left( \frac{180^\circ}{\pi} \right) \approx 117^\circ. \quad \text{Convert } \theta = \frac{125\pi}{192} \text{ to degrees.}$$

Now try Exercise 63.

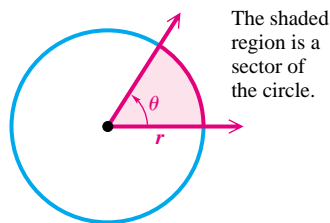


Figure 8

The shaded region is a sector of the circle.

**Area of a Sector of a Circle** A **sector of a circle** is the portion of the interior of a circle intercepted by a central angle. Think of it as a “piece of pie.” See Figure 8. A complete circle can be thought of as an angle with measure  $2\pi$  radians. If a central angle for a sector has measure  $\theta$  radians, then the sector makes up the fraction  $\frac{\theta}{2\pi}$  of a complete circle. The area of a complete circle with radius  $r$  is  $A = \pi r^2$ . Therefore,

$$\text{area of the sector} = \frac{\theta}{2\pi} (\pi r^2) = \frac{1}{2} r^2 \theta, \quad \theta \text{ in radians.}$$

This discussion is summarized as follows.

### Area of a Sector

The area of a sector of a circle of radius  $r$  and central angle  $\theta$  is given by

$$A = \frac{1}{2}r^2\theta, \quad \theta \text{ in radians.}$$

**CAUTION** As in the formula for arc length, the value of  $\theta$  must be in radians when using this formula for the area of a sector.

### EXAMPLE 7 Finding the Area of a Sector-Shaped Field

Find the area of the sector-shaped field shown in Figure 9.

**Solution** First, convert  $15^\circ$  to radians.

$$15^\circ = 15 \left( \frac{\pi}{180} \right) = \frac{\pi}{12} \text{ radian}$$

Now use the formula to find the area of a sector of a circle with radius  $r = 321$ .

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(321)^2 \left( \frac{\pi}{12} \right) \approx 13,500 \text{ m}^2$$

Now try Exercise 77.



Figure 9

## 6.1 Exercises

1.  $\frac{\pi}{3}$  2.  $\frac{\pi}{2}$  3.  $\frac{5\pi}{6}$  4.  $\frac{3\pi}{2}$
5.  $\frac{7\pi}{4}$  6.  $\frac{8\pi}{3}$  7.  $-\frac{\pi}{4}$
8.  $-\frac{7\pi}{6}$  9. .68 10. 1.29
11. 2.43 12. 3.05 13. 1.122
14. 2.140 15. 1 16. 2

Convert each degree measure to radians. Leave answers as multiples of  $\pi$ . See Example 1(a).

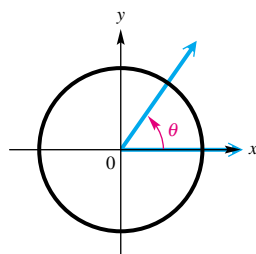
1.  $60^\circ$  2.  $90^\circ$  3.  $150^\circ$  4.  $270^\circ$
5.  $315^\circ$  6.  $480^\circ$  7.  $-45^\circ$  8.  $-210^\circ$

Convert each degree measure to radians. See Example 1(b).

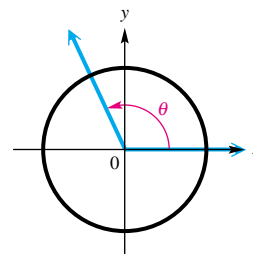
9.  $39^\circ$  10.  $74^\circ$  11.  $139^\circ 10'$
12.  $174^\circ 50'$  13.  $64.29^\circ$  14.  $122.62^\circ$

**Concept Check** In Exercises 15–18, each angle  $\theta$  is an integer when measured in radians. Give the radian measure of the angle.

15.



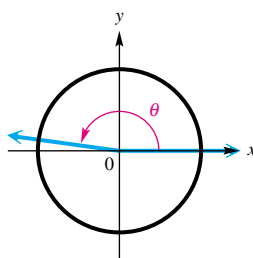
16.



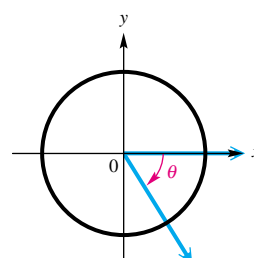
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17. 3   18. -1   21.  $60^\circ$   
 22.  $480^\circ$    23.  $315^\circ$    24.  $120^\circ$   
 25.  $330^\circ$    26.  $675^\circ$    27.  $-30^\circ$   
 28.  $-63^\circ$    29.  $114.6^\circ$   
 30.  $286.5^\circ$    31.  $99.7^\circ$    32.  $19.6^\circ$   
 33.  $-564.2^\circ$    34.  $-198.9^\circ$   
 35. 1   36.  $\sqrt{2}$    37.  $-\frac{\sqrt{3}}{3}$   
 38.  $\frac{1}{2}$    39. -1   40.  $\frac{1}{2}$   
 41.  $\frac{\sqrt{3}}{2}$    42. -1

17.



18.



19. In your own words, explain how to convert  
 (a) degree measure to radian measure;   (b) radian measure to degree measure.  
 20. Explain the difference between degree measure and radian measure.

Convert each radian measure to degrees. See Example 2(a).

21.  $\frac{\pi}{3}$    22.  $\frac{8\pi}{3}$    23.  $\frac{7\pi}{4}$    24.  $\frac{2\pi}{3}$   
 25.  $\frac{11\pi}{6}$    26.  $\frac{15\pi}{4}$    27.  $-\frac{\pi}{6}$    28.  $-\frac{7\pi}{20}$

Convert each radian measure to degrees. Give answers using decimal degrees to the nearest tenth. See Example 2(b).

29. 2   30. 5   31. 1.74  
 32. .3417   33. -9.84763   34. -3.47189

## Relating Concepts

For individual or collaborative investigation  
 (Exercises 35–42)

In anticipation of the material in the next section, we show how to find the trigonometric function values of radian-measured angles. Suppose we want to find  $\sin \frac{5\pi}{6}$ . One way to do this is to convert  $\frac{5\pi}{6}$  radians to  $150^\circ$ , and then use the methods of Chapter 5 to evaluate:

$$\sin \frac{5\pi}{6} = \sin 150^\circ = + \sin 30^\circ = \frac{1}{2}. \quad (\text{Section 5.3})$$

Sine is positive in quadrant II.   Reference angle for  $150^\circ$

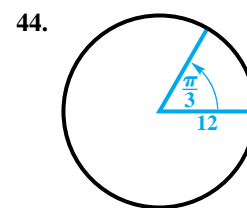
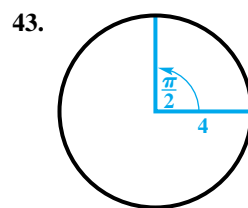
Use this technique to find each function value. Give exact values.

35.  $\tan \frac{\pi}{4}$    36.  $\csc \frac{\pi}{4}$    37.  $\cot \frac{2\pi}{3}$    38.  $\cos \frac{\pi}{3}$   
 39.  $\sec \pi$    40.  $\sin \left(-\frac{7\pi}{6}\right)$    41.  $\cos \left(-\frac{\pi}{6}\right)$    42.  $\tan \left(-\frac{9\pi}{4}\right)$

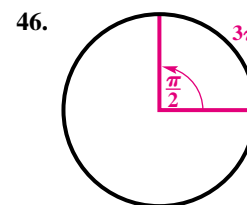
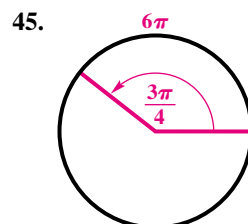


43.  $2\pi$  44.  $4\pi$  45. 8 46. 6  
 47. 1 48. 1.5 49. 25.8 cm  
 50. 3.08 cm 51. 5.05 m  
 52. 169 cm 53. The length is  
 doubled. 54.  $s = \frac{\pi r \theta}{180}$   
 55. 3500 km 56. 1500 km  
 57. 5900 km 58. 8800 km

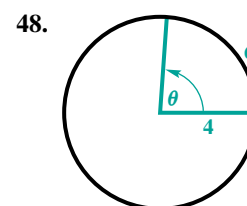
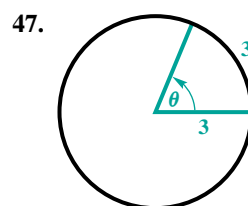
**Concept Check** Find the exact length of each arc intercepted by the given central angle.



**Concept Check** Find the radius of each circle.



**Concept Check** Find the measure of each central angle (in radians).



**Unless otherwise directed, give calculator approximations in your answers in the rest of this exercise set.**

Find the length of each arc intercepted by a central angle  $\theta$  in a circle of radius  $r$ . See Example 3.

49.  $r = 12.3$  cm,  $\theta = \frac{2\pi}{3}$  radians

50.  $r = .892$  cm,  $\theta = \frac{11\pi}{10}$  radians

51.  $r = 4.82$  m,  $\theta = 60^\circ$

52.  $r = 71.9$  cm,  $\theta = 135^\circ$

53. **Concept Check** If the radius of a circle is doubled, how is the length of the arc intercepted by a fixed central angle changed?

54. **Concept Check** Radian measure simplifies many formulas, such as the formula for arc length,  $s = r\theta$ . Give the corresponding formula when  $\theta$  is measured in degrees instead of radians.

**Distance Between Cities** Find the distance in kilometers between each pair of cities, assuming they lie on the same north-south line. See Example 4.

55. Panama City, Panama,  $9^\circ$  N, and Pittsburgh, Pennsylvania,  $40^\circ$  N

56. Farmersville, California,  $36^\circ$  N, and Penticton, British Columbia,  $49^\circ$  N

57. New York City, New York,  $41^\circ$  N, and Lima, Peru,  $12^\circ$  S

58. Halifax, Nova Scotia,  $45^\circ$  N, and Buenos Aires, Argentina,  $34^\circ$  S

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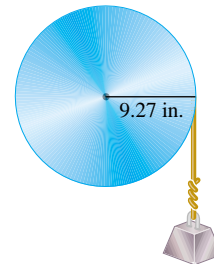
59.  $44^\circ \text{ N}$  60.  $43^\circ \text{ N}$   
 61. (a) 11.6 in. (b)  $37^\circ 5'$   
 62. 12.7 cm 63.  $38.5^\circ$   
 64. 18.7 cm 65. 146 in.  
 66. (a) 39,616 rotations

59. **Latitude of Madison** Madison, South Dakota, and Dallas, Texas, are 1200 km apart and lie on the same north-south line. The latitude of Dallas is  $33^\circ \text{ N}$ . What is the latitude of Madison?
60. **Latitude of Toronto** Charleston, South Carolina, and Toronto, Canada, are 1100 km apart and lie on the same north-south line. The latitude of Charleston is  $33^\circ \text{ N}$ . What is the latitude of Toronto?

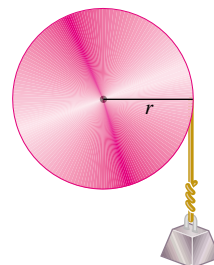
Work each problem. See Examples 5 and 6.

61. **Pulley Raising a Weight**

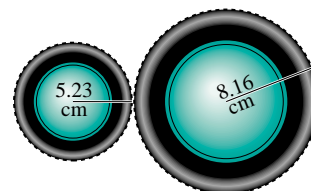
- (a) How many inches will the weight in the figure rise if the pulley is rotated through an angle of  $71^\circ 50'$ ?
- (b) Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in.?



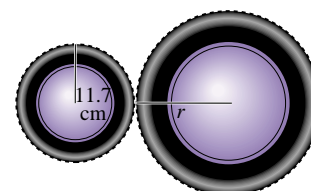
62. **Pulley Raising a Weight** Find the radius of the pulley in the figure if a rotation of  $51.6^\circ$  raises the weight 11.4 cm.



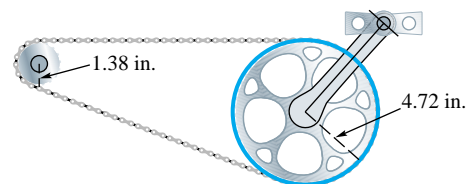
63. **Rotating Wheels** The rotation of the smaller wheel in the figure causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through  $60.0^\circ$ ?



64. **Rotating Wheels** Find the radius of the larger wheel in the figure if the smaller wheel rotates  $80.0^\circ$  when the larger wheel rotates  $50.0^\circ$ .



65. **Bicycle Chain Drive** The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through  $180^\circ$ ? Assume the radius of the bicycle wheel is 13.6 in.



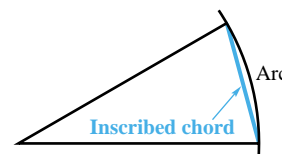
66. **Pickup Truck Speedometer** The speedometer of Terry's small pickup truck is designed to be accurate with tires of radius 14 in.
- (a) Find the number of rotations of a tire in 1 hr if the truck is driven at 55 mph.

66. (b) 62.9 mi; yes  
 67. .20 km 68. 850 ft  
 69.  $6\pi$  70.  $16\pi$  71. 1.5  
 72. 1 73.  $1116.1 \text{ m}^2$   
 74.  $3744.8 \text{ km}^2$  75.  $706.9 \text{ ft}^2$   
 76.  $10,602.9 \text{ yd}^2$  77.  $114.0 \text{ cm}^2$   
 78.  $365.3 \text{ m}^2$  79.  $1885.0 \text{ mi}^2$   
 80.  $19,085.2 \text{ km}^2$  81. 3.6  
 82. 16 m

- (b) Suppose that oversized tires of radius 16 in. are placed on the truck. If the truck is now driven for 1 hr with the speedometer reading 55 mph, how far has the truck gone? If the speed limit is 55 mph, does Terry deserve a speeding ticket?

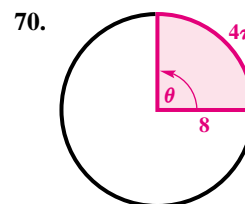
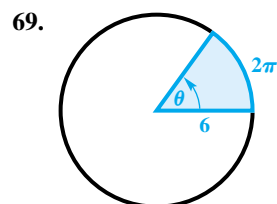
If a central angle is very small, there is little difference in length between an arc and the inscribed chord. See the figure. Approximate each of the following lengths by finding the necessary arc length. (Note: When a central angle intercepts an arc, the arc is said to **subtend** the angle.)

Arc length  $\approx$  length of inscribed chord

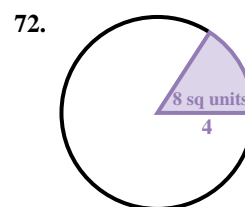
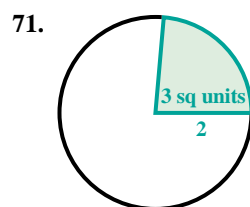


67. **Length of a Train** A railroad track in the desert is 3.5 km away. A train on the track subtends (horizontally) an angle of  $3^\circ 20'$ . Find the length of the train.  
 68. **Distance to a Boat** The mast of Brent Simon's boat is 32 ft high. If it subtends an angle of  $2^\circ 10'$ , how far away is it?

**Concept Check** Find the area of each sector.



**Concept Check** Find the measure (in radians) of each central angle. The number inside the sector is the area.



Find the area of a sector of a circle having radius  $r$  and central angle  $\theta$ . See Example 7.

73.  $r = 29.2 \text{ m}$ ,  $\theta = \frac{5\pi}{6}$  radians  
 74.  $r = 59.8 \text{ km}$ ,  $\theta = \frac{2\pi}{3}$  radians  
 75.  $r = 30.0 \text{ ft}$ ,  $\theta = \frac{\pi}{2}$  radians  
 76.  $r = 90.0 \text{ yd}$ ,  $\theta = \frac{5\pi}{6}$  radians  
 77.  $r = 12.7 \text{ cm}$ ,  $\theta = 81^\circ$   
 78.  $r = 18.3 \text{ m}$ ,  $\theta = 125^\circ$   
 79.  $r = 40.0 \text{ mi}$ ,  $\theta = 135^\circ$   
 80.  $r = 90.0 \text{ km}$ ,  $\theta = 270^\circ$

Work each problem.

81. Find the measure (in radians) of a central angle of a sector of area  $16 \text{ in.}^2$  in a circle of radius 3.0 in.  
 82. Find the radius of a circle in which a central angle of  $\frac{\pi}{6}$  radian determines a sector of area  $64 \text{ m}^2$ .

## 544 CHAPTER 6 The Circular Functions and Their Graphs

83. (a)  $13\frac{1}{3}^\circ; \frac{2\pi}{27}$  (b) 480 ft

(c)  $\frac{160}{9} \approx 17.8$  ft

(d) approximately 672 ft<sup>2</sup>

84. 75.4 in.<sup>2</sup>

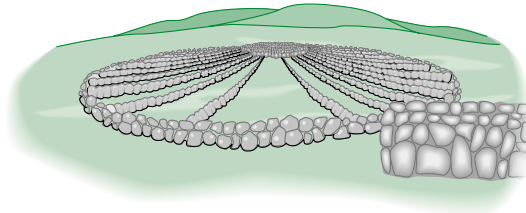
85. (a) 140 ft (b) 102 ft

(c) 622 ft<sup>2</sup>

86. (a) 550 m (b) 1800 m

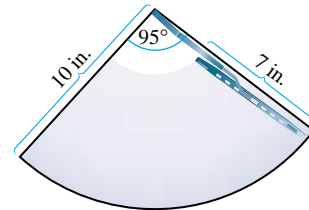
87. 1900 yd<sup>2</sup> 88. 1.15 mi

83. **Measures of a Structure** The figure shows Medicine Wheel, a Native American structure in northern Wyoming. This circular structure is perhaps 2500 yr old. There are 27 aboriginal spokes in the wheel, all equally spaced.



- (a) Find the measure of each central angle in degrees and in radians.  
 (b) If the radius of the wheel is 76 ft, find the circumference.  
 (c) Find the length of each arc intercepted by consecutive pairs of spokes.  
 (d) Find the area of each sector formed by consecutive spokes.

84. **Area Cleaned by a Windshield Wiper** The Ford Model A, built from 1928 to 1931, had a single windshield wiper on the driver's side. The total arm and blade was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



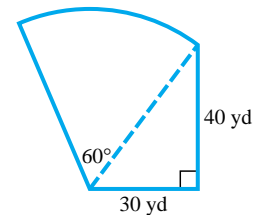
85. **Circular Railroad Curves** In the United States, circular railroad curves are designated by the *degree of curvature*, the central angle subtended by a chord of 100 ft. Suppose a portion of track has curvature 42°. (Source: Hay, W., *Railroad Engineering*, John Wiley & Sons, 1982.)

- (a) What is the radius of the curve?  
 (b) What is the length of the arc determined by the 100-ft chord?  
 (c) What is the area of the portion of the circle bounded by the arc and the 100-ft chord?

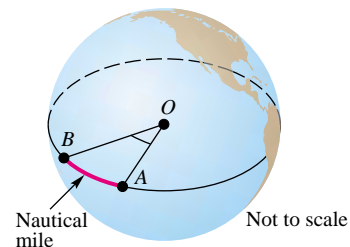
86. **Land Required for a Solar-Power Plant** A 300-megawatt solar-power plant requires approximately 950,000 m<sup>2</sup> of land area in order to collect the required amount of energy from sunlight.

- (a) If this land area is circular, what is its radius?  
 (b) If this land area is a 35° sector of a circle, what is its radius?

87. **Area of a Lot** A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of a circle. Find the area of the lot shown in the figure. (Source: Anderson, J. and E. Michael, *Introduction to Surveying*, McGraw-Hill, 1985.)

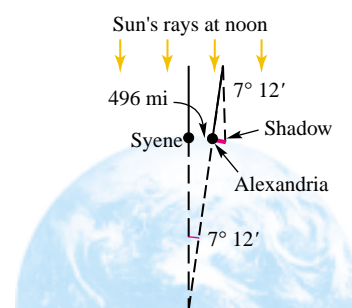


88. **Nautical Miles** Nautical miles are used by ships and airplanes. They are different from *statute miles*, which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle  $AOB$  of 1 min, as illustrated in the figure. If the equatorial radius of Earth is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile. Round your answer to two decimal places.

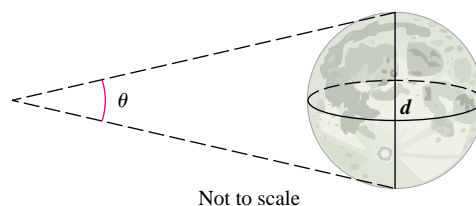


89. radius: 3947 mi;  
circumference: 24,800 mi  
90. approximately 2156 mi  
91. The area is quadrupled.  
92.  $A = \frac{\pi r^2 \theta}{360}$

**89. Circumference of Earth** The first accurate estimate of the distance around Earth was done by the Greek astronomer Eratosthenes (276–195 B.C.), who noted that the noontime position of the sun at the summer solstice differed by  $7^\circ 12'$  from the city of Syene to the city of Alexandria. (See the figure.) The distance between these two cities is 496 mi. Use the arc length formula to estimate the radius of Earth. Then find the circumference of Earth. (Source: Zeilik, M., *Introductory Astronomy and Astrophysics*, Third Edition, Saunders College Publishers, 1992.)



**90. Diameter of the Moon** The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter  $d$  of the moon if angle  $\theta$  in the figure is measured to be  $.517^\circ$ .



91. **Concept Check** If the radius of a circle is doubled and the central angle of a sector is unchanged, how is the area of the sector changed?  
92. **Concept Check** Give the corresponding formula for the area of a sector when the angle is measured in degrees.

## 6.2 The Unit Circle and Circular Functions

Circular Functions ■ Finding Values of Circular Functions ■ Determining a Number with a Given Circular Function Value ■ Angular and Linear Speed

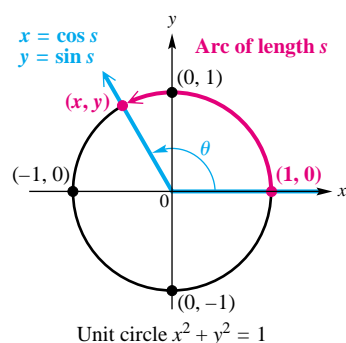


Figure 10

In Section 5.2, we defined the six trigonometric functions in such a way that the domain of each function was a set of *angles* in standard position. These angles can be measured in degrees or in radians. In advanced courses, such as calculus, it is necessary to modify the trigonometric functions so that their domains consist of *real numbers* rather than angles. We do this by using the relationship between an angle  $\theta$  and an arc of length  $s$  on a circle.

**Circular Functions** In Figure 10, we start at the point  $(1,0)$  and measure an arc of length  $s$  along the circle. If  $s > 0$ , then the arc is measured in a counter-clockwise direction, and if  $s < 0$ , then the direction is clockwise. (If  $s = 0$ , then no arc is measured.) Let the endpoint of this arc be at the point  $(x, y)$ . The circle in Figure 10 is a **unit circle**—it has center at the origin and radius 1 unit (hence the name *unit circle*). Recall from algebra that the equation of this circle is

$$x^2 + y^2 = 1. \quad (\text{Section 2.1})$$

We saw in the previous section that the radian measure of  $\theta$  is related to the arc length  $s$ . In fact, for  $\theta$  measured in radians, we know that  $s = r\theta$ . Here,  $r = 1$ , so  $s$ , which is measured in linear units such as inches or centimeters, is numerically equal to  $\theta$ , measured in radians. Thus, the trigonometric functions of angle  $\theta$  in radians found by choosing a point  $(x, y)$  on the unit circle can be rewritten as functions of the arc length  $s$ , a real number. When interpreted this way, they are called **circular functions**.

### Looking Ahead to Calculus

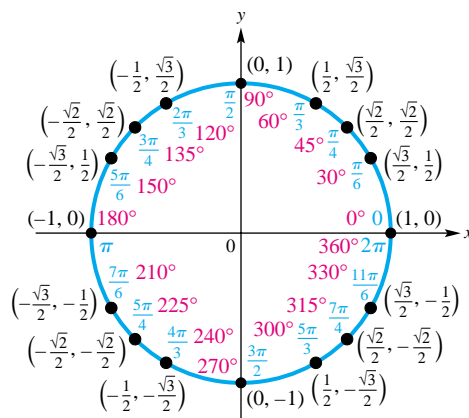
If you plan to study calculus, you must become very familiar with radian measure. In calculus, the trigonometric or circular functions are always understood to have real number domains.

### Circular Functions

$\sin s = y$	$\cos s = x$	$\tan s = \frac{y}{x} \quad (x \neq 0)$
$\csc s = \frac{1}{y} \quad (y \neq 0)$	$\sec s = \frac{1}{x} \quad (x \neq 0)$	$\cot s = \frac{x}{y} \quad (y \neq 0)$

Since  $x$  represents the cosine of  $s$  and  $y$  represents the sine of  $s$ , and because of the discussion in Section 6.1 on converting between degrees and radians, we can summarize a great deal of information in a concise manner, as seen in Figure 11.\*

**TEACHING TIP** Draw a unit circle on the chalkboard and trace a point slowly counterclockwise, starting at  $(1, 0)$ . Students should see that the  $y$ -value (sine function) starts at 0, increases to 1 (at  $\frac{\pi}{2}$  radians), decreases to 0 (at  $\pi$  radians), decreases again to  $-1$  (at  $\frac{3\pi}{2}$  radians), and returns to 0 after one full revolution ( $2\pi$  radians). This type of analysis will help illustrate the range associated with the sine and cosine functions.



Unit circle  $x^2 + y^2 = 1$

Figure 11

**NOTE** Since  $\sin s = y$  and  $\cos s = x$ , we can replace  $x$  and  $y$  in the equation  $x^2 + y^2 = 1$  and obtain the Pythagorean identity

$$\cos^2 s + \sin^2 s = 1.$$

The ordered pair  $(x, y)$  represents a point on the unit circle, and therefore

$$-1 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1,$$

so  $-1 \leq \cos s \leq 1 \quad \text{and} \quad -1 \leq \sin s \leq 1.$

\*The authors thank Professor Marvel Townsend of the University of Florida for her suggestion to include this figure.

For any value of  $s$ , both  $\sin s$  and  $\cos s$  exist, so the domain of these functions is the set of all real numbers. For  $\tan s$ , defined as  $\frac{y}{x}$ ,  $x$  must not equal 0. The only way  $x$  can equal 0 is when the arc length  $s$  is  $\frac{\pi}{2}$ ,  $-\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$ ,  $-\frac{3\pi}{2}$ , and so on. To avoid a 0 denominator, the domain of the tangent function must be restricted to those values of  $s$  satisfying

$$s \neq (2n + 1)\frac{\pi}{2}, \quad n \text{ any integer.}$$

The definition of secant also has  $x$  in the denominator, so the domain of secant is the same as the domain of tangent. Both cotangent and cosecant are defined with a denominator of  $y$ . To guarantee that  $y \neq 0$ , the domain of these functions must be the set of all values of  $s$  satisfying

$$s \neq n\pi, \quad n \text{ any integer.}$$

In summary, the domains of the circular functions are as follows.

### Domains of the Circular Functions

Assume that  $n$  is any integer and  $s$  is a real number.

**Sine and Cosine Functions:**  $(-\infty, \infty)$

**Tangent and Secant Functions:**  $\left\{s \mid s \neq (2n + 1)\frac{\pi}{2}\right\}$

**Cotangent and Cosecant Functions:**  $\{s \mid s \neq n\pi\}$

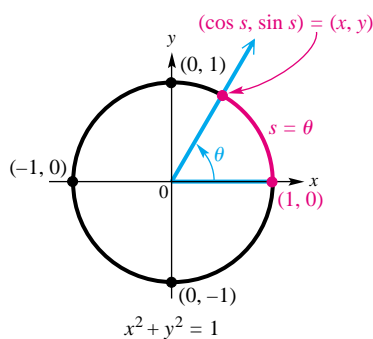


Figure 12

**Finding Values of Circular Functions** The circular functions (functions of real numbers) are closely related to the trigonometric functions of angles measured in radians. To see this, let us assume that angle  $\theta$  is in standard position, superimposed on the unit circle, as shown in Figure 12. Suppose further that  $\theta$  is the radian measure of this angle. Using the arc length formula  $s = r\theta$  with  $r = 1$ , we have  $s = \theta$ . Thus, the length of the intercepted arc is the real number that corresponds to the radian measure of  $\theta$ . Using the definitions of the trigonometric functions, we have

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y = \sin s, \quad \text{and} \quad \cos \theta = \frac{x}{r} = \frac{x}{1} = x = \cos s,$$

and so on. As shown here, the trigonometric functions and the circular functions lead to the same function values, provided we think of the angles as being in radian measure. This leads to the following important result concerning evaluation of circular functions.

### Evaluating a Circular Function

Circular function values of real numbers are obtained in the same manner as trigonometric function values of angles measured in radians. This applies both to methods of finding exact values (such as reference angle analysis) and to calculator approximations. Calculators must be in radian mode when finding circular function values.

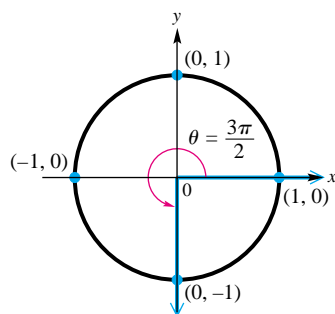


Figure 13

**TEACHING TIP** Point out the importance of being able to sketch the unit circle as in Figure 13. This will enable students to more easily determine function values of multiples of  $\frac{\pi}{2}$ .

**EXAMPLE 1** Finding Exact Circular Function Values

Find the exact values of  $\sin \frac{3\pi}{2}$ ,  $\cos \frac{3\pi}{2}$ , and  $\tan \frac{3\pi}{2}$ .

**Solution** Evaluating a circular function at the real number  $\frac{3\pi}{2}$  is equivalent to evaluating it at  $\frac{3\pi}{2}$  radians. An angle of  $\frac{3\pi}{2}$  radians intersects the unit circle at the point  $(0, -1)$ , as shown in Figure 13. Since

$$\sin s = y, \quad \cos s = x, \quad \text{and} \quad \tan s = \frac{y}{x},$$

it follows that

$$\sin \frac{3\pi}{2} = -1, \quad \cos \frac{3\pi}{2} = 0, \quad \text{and} \quad \tan \frac{3\pi}{2} \text{ is undefined.}$$

Now try Exercise 1.

**EXAMPLE 2** Finding Exact Circular Function Values

- Use Figure 11 to find the exact values of  $\cos \frac{7\pi}{4}$  and  $\sin \frac{7\pi}{4}$ .
- Use Figure 11 to find the exact value of  $\tan\left(-\frac{5\pi}{3}\right)$ .
- Use reference angles and degree/radian conversion to find the exact value of  $\cos \frac{2\pi}{3}$ .

**Solution**

- In Figure 11, we see that the terminal side of  $\frac{7\pi}{4}$  radians intersects the unit circle at  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ . Thus,

$$\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}.$$

- Angles of  $-\frac{5\pi}{3}$  radians and  $\frac{\pi}{3}$  radians are coterminal. Their terminal sides intersect the unit circle at  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ , so

$$\tan\left(-\frac{5\pi}{3}\right) = \tan \frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

- An angle of  $\frac{2\pi}{3}$  radians corresponds to an angle of  $120^\circ$ . In standard position,  $120^\circ$  lies in quadrant II with a reference angle of  $60^\circ$ , so

$$\cos \frac{2\pi}{3} = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}.$$

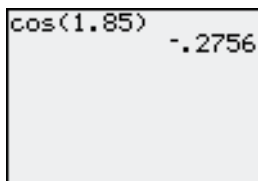
Cosine is negative in quadrant II.

Reference angle (Section 5.3)

Now try Exercises 7, 17, and 21.

**NOTE** Examples 1 and 2 illustrate that there are several methods of finding exact circular function values.





Radian mode

This is how a calculator displays the result of Example 3(a), correct to four decimal digits.

**TEACHING TIP** Remind students *not* to use the  $\sin^{-1}$ ,  $\cos^{-1}$ , and  $\tan^{-1}$  keys when evaluating reciprocal functions with a calculator.

### EXAMPLE 3 Approximating Circular Function Values

Find a calculator approximation to four decimal places for each circular function value.

- (a)  $\cos 1.85$       (b)  $\cos .5149$       (c)  $\cot 1.3209$       (d)  $\sec(-2.9234)$

#### Solution

(a) With a calculator in radian mode, we find  $\cos 1.85 \approx -.2756$ .

(b)  $\cos .5149 \approx .8703$  Use a calculator in radian mode.

(c) As before, to find cotangent, secant, and cosecant function values, we must use the appropriate reciprocal functions. To find  $\cot 1.3209$ , first find  $\tan 1.3209$  and then find the reciprocal.

$$\cot 1.3209 = \frac{1}{\tan 1.3209} \approx .2552$$

$$(d) \sec(-2.9234) = \frac{1}{\cos(-2.9234)} \approx -1.0243$$

Now try Exercises 23, 29, and 33.

**CAUTION** A common error in trigonometry is using calculators in degree mode when radian mode should be used. Remember, *if you are finding a circular function value of a real number, the calculator must be in radian mode.*

**Determining a Number with a Given Circular Function Value** Recall from Section 5.3 how we used a calculator to determine an angle measure, given a trigonometric function value of the angle.

### EXAMPLE 4 Finding a Number Given Its Circular Function Value

- (a) Approximate the value of  $s$  in the interval  $[0, \frac{\pi}{2}]$ , if  $\cos s = .9685$ .  
 (b) Find the exact value of  $s$  in the interval  $[\pi, \frac{3\pi}{2}]$ , if  $\tan s = 1$ .

#### Solution

(a) Since we are given a cosine value and want to determine the real number in  $[0, \frac{\pi}{2}]$  having this cosine value, we use the *inverse cosine* function of a calculator. With the calculator in radian mode, we find

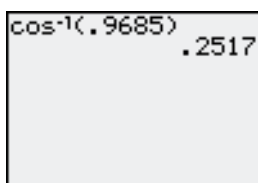
$$\cos^{-1}(.9685) \approx .2517. \quad (\text{Section 5.3})$$

See Figure 14. (Refer to your owner's manual to determine how to evaluate the  $\sin^{-1}$ ,  $\cos^{-1}$ , and  $\tan^{-1}$  functions with your calculator.)

(b) Recall that  $\tan \frac{\pi}{4} = 1$ , and in quadrant III  $\tan s$  is positive. Therefore,

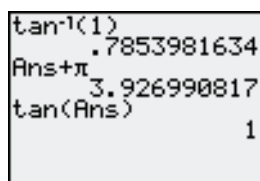
$$\tan\left(\pi + \frac{\pi}{4}\right) = \tan \frac{5\pi}{4} = 1,$$

and  $s = \frac{5\pi}{4}$ . Figure 15 supports this result.



The calculator is set to show four decimal digits in the answer.

Figure 14



This screen supports the result of Example 4(b). The calculator is in radian mode.

Figure 15

Now try Exercises 49 and 55.

**CONNECTIONS** A convenient way to see the sine, cosine, and tangent trigonometric ratios geometrically is shown in Figure 16 for  $\theta$  in quadrants I and II. The circle shown is the unit circle, which has radius 1. By remembering this figure and the segments that represent the sine, cosine, and tangent functions, you can quickly recall properties of the trigonometric functions. Horizontal line segments to the left of the origin and vertical line segments below the  $x$ -axis represent negative values. Note that the tangent line must be tangent to the circle at  $(1, 0)$ , for any quadrant in which  $\theta$  lies.

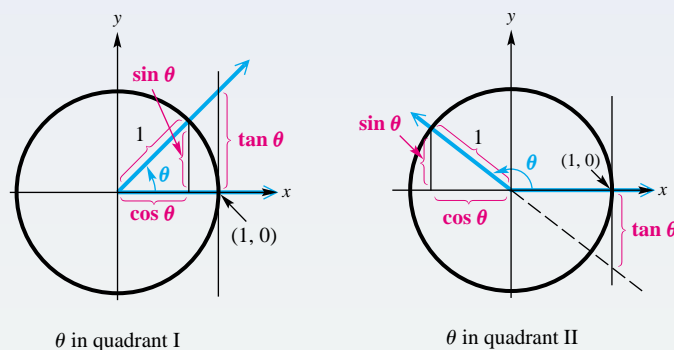


Figure 16

### For Discussion or Writing

See Figure 17. Use the definition of the trigonometric functions and similar triangles to show that  $PQ = \sin \theta$ ,  $OQ = \cos \theta$ , and  $AB = \tan \theta$ .

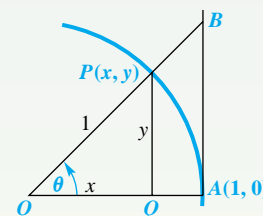


Figure 17

$$PQ = y = \frac{y}{1} = \sin \theta;$$

$$OQ = x = \frac{x}{1} = \cos \theta;$$

$$AB = \frac{AB}{1} = \frac{AB}{AO} = \frac{y}{x}$$

(by similar triangles)  $= \tan \theta$



**Angular and Linear Speed** The human joint that can be flexed the fastest is the wrist, which can rotate through  $90^\circ$ , or  $\frac{\pi}{2}$  radians, in .045 sec while holding a tennis racket. **Angular speed**  $\omega$  (omega) measures the speed of rotation and is defined by

$$\omega = \frac{\theta}{t},$$

where  $\theta$  is the angle of rotation in radians and  $t$  is time. The angular speed of a human wrist swinging a tennis racket is

$$\omega = \frac{\theta}{t} = \frac{\frac{\pi}{2}}{.045} \approx 35 \text{ radians per sec.}$$

The **linear speed**  $v$  at which the tip of the racket travels as a result of flexing the wrist is given by

$$v = r\omega,$$

where  $r$  is the radius (distance) from the tip of the racket to the wrist joint. If  $r = 2$  ft, then the speed at the tip of the racket is

$$v = r\omega \approx 2(35) = 70 \text{ ft per sec,} \quad \text{or about 48 mph.}$$

In a tennis serve the arm rotates at the shoulder, so the final speed of the racket is considerably faster. (Source: Cooper, J. and R. Glassow, *Kinesiology*, Second Edition, C.V. Mosby, 1968.)

### EXAMPLE 5 Finding Angular Speed of a Pulley and Linear Speed of a Belt

A belt runs a pulley of radius 6 cm at 80 revolutions per min.

- (a) Find the angular speed of the pulley in radians per second.  
 (b) Find the linear speed of the belt in centimeters per second.

#### Solution

- (a) In 1 min, the pulley makes 80 revolutions. Each revolution is  $2\pi$  radians, for a total of

$$80(2\pi) = 160\pi \text{ radians per min.}$$

Since there are 60 sec in 1 min, we find  $\omega$ , the angular speed in radians per second, by dividing  $160\pi$  by 60.

$$\omega = \frac{160\pi}{60} = \frac{8\pi}{3} \text{ radians per sec}$$

- (b) The linear speed of the belt will be the same as that of a point on the circumference of the pulley. Thus,

$$v = r\omega = 6\left(\frac{8\pi}{3}\right) = 16\pi \approx 50.3 \text{ cm per sec.}$$

Now try Exercise 95.

Suppose that an object is moving at a constant speed. If it travels a distance  $s$  in time  $t$ , then its linear speed  $v$  is given by

$$v = \frac{s}{t}.$$

If the object is traveling in a circle, then  $s = r\theta$ , where  $r$  is the radius of the circle and  $\theta$  is the angle of rotation. Thus, we can write

$$v = \frac{r\theta}{t}.$$

The formulas for angular and linear speed are summarized in the table.

Angular Speed	Linear Speed
$\omega = \frac{\theta}{t}$	$v = \frac{s}{t}$
( $\omega$ in radians per unit time, $\theta$ in radians)	$v = \frac{r\theta}{t}$
	$v = r\omega$

**TEACHING TIP** Mention that, while problems involving linear and angular speed can be solved using the formula for circumference, the formulas presented in this section are much more efficient.

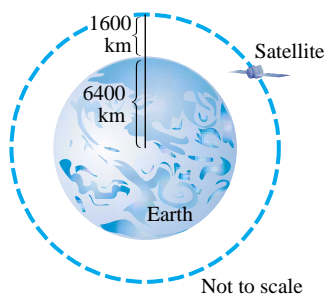


Figure 18

**EXAMPLE 6** Finding Linear Speed and Distance Traveled by a Satellite

A satellite traveling in a circular orbit 1600 km above the surface of Earth takes 2 hr to make an orbit. The radius of Earth is 6400 km. See Figure 18.

- (a) Find the linear speed of the satellite.  
 (b) Find the distance the satellite travels in 4.5 hr.

**Solution**

- (a) The distance of the satellite from the center of Earth is

$$r = 1600 + 6400 = 8000 \text{ km.}$$

For one orbit,  $\theta = 2\pi$ , and

$$s = r\theta = 8000(2\pi) \text{ km. (Section 6.1)}$$

Since it takes 2 hr to complete an orbit, the linear speed is

$$v = \frac{s}{t} = \frac{8000(2\pi)}{2} = 8000\pi \approx 25,000 \text{ km per hr.}$$

- (b)  $s = vt = 8000\pi(4.5) = 36,000\pi \approx 110,000 \text{ km}$

Now try Exercise 93.

**6.2 Exercises**

1. (a) 1 (b) 0 (c) undefined

2. (a) 0 (b) -1 (c) 0

3. (a) 0 (b) 1 (c) 0

4. (a) 0 (b) -1 (c) 0

5. (a) 0 (b) -1 (c) 0

6. (a) 1 (b) 0 (c) undefined

7.  $-\frac{1}{2}$  8.  $\frac{1}{2}$  9. -1 10. -2

11. -2 12.  $-\sqrt{3}$  13.  $-\frac{1}{2}$

14.  $-\sqrt{3}$  15.  $\frac{\sqrt{2}}{2}$  16.  $-\sqrt{2}$

17.  $\frac{\sqrt{3}}{2}$  18.  $-\frac{1}{2}$  19.  $\frac{2\sqrt{3}}{3}$

20.  $\frac{2\sqrt{3}}{3}$  21.  $-\frac{\sqrt{3}}{3}$  22.  $-\frac{\sqrt{2}}{2}$

23. .5736 24. .7314 25. .4068

26. .5397 27. 1.2065

28. .1944 29. 14.3338

30. 1.0170 31. -1.0460

32. -2.1291 33. -3.8665

34. 1.1848

For each value of  $\theta$ , find (a)  $\sin \theta$ , (b)  $\cos \theta$ , and (c)  $\tan \theta$ . See Example 1.

1.  $\theta = \frac{\pi}{2}$

2.  $\theta = \pi$

3.  $\theta = 2\pi$

4.  $\theta = 3\pi$

5.  $\theta = -\pi$

6.  $\theta = -\frac{3\pi}{2}$

Find the exact circular function value for each of the following. See Example 2.

7.  $\sin \frac{7\pi}{6}$

8.  $\cos \frac{5\pi}{3}$

9.  $\tan \frac{3\pi}{4}$

10.  $\sec \frac{2\pi}{3}$

11.  $\csc \frac{11\pi}{6}$

12.  $\cot \frac{5\pi}{6}$

13.  $\cos\left(-\frac{4\pi}{3}\right)$

14.  $\tan \frac{17\pi}{3}$

15.  $\cos \frac{7\pi}{4}$

16.  $\sec \frac{5\pi}{4}$

17.  $\sin\left(-\frac{4\pi}{3}\right)$

18.  $\sin\left(-\frac{5\pi}{6}\right)$

19.  $\sec \frac{23\pi}{6}$

20.  $\csc \frac{13\pi}{3}$

21.  $\tan \frac{5\pi}{6}$

22.  $\cos \frac{3\pi}{4}$

Find a calculator approximation for each circular function value. See Example 3.

23.  $\sin .6109$

24.  $\sin .8203$

25.  $\cos(-1.1519)$

26.  $\cos(-5.2825)$

27.  $\tan 4.0203$

28.  $\tan 6.4752$

29.  $\csc(-9.4946)$

30.  $\csc 1.3875$

31.  $\sec 2.8440$

32.  $\sec(-8.3429)$

33.  $\cot 6.0301$

34.  $\cot 3.8426$

35. .7 36.  $-.75$  37. 4

38. 4.4 39. negative

40. negative 41. negative

42. positive 43. positive

44. negative

45.  $\sin \theta = \frac{\sqrt{2}}{2}$ ;  $\cos \theta = \frac{\sqrt{2}}{2}$ ;

$\tan \theta = 1$ ;  $\cot \theta = 1$ ;

$\sec \theta = \sqrt{2}$ ;  $\csc \theta = \sqrt{2}$

46.  $\sin \theta = \frac{8}{17}$ ;  $\cos \theta = -\frac{15}{17}$ ;

$\tan \theta = -\frac{8}{15}$ ;  $\cot \theta = -\frac{15}{8}$ ;

$\sec \theta = -\frac{17}{15}$ ;  $\csc \theta = \frac{17}{8}$

47.  $\sin \theta = -\frac{12}{13}$ ;  $\cos \theta = \frac{5}{13}$ ;

$\tan \theta = -\frac{12}{5}$ ;  $\cot \theta = -\frac{5}{12}$ ;

$\sec \theta = \frac{13}{5}$ ;  $\csc \theta = -\frac{13}{12}$

48.  $\sin \theta = -\frac{1}{2}$ ;  $\cos \theta = -\frac{\sqrt{3}}{2}$ ;

$\tan \theta = \frac{\sqrt{3}}{3}$ ;  $\cot \theta = \sqrt{3}$ ;

$\sec \theta = -\frac{2\sqrt{3}}{3}$ ;  $\csc \theta = -2$

49. .2095 50. .6720

51. 1.4426 52. 1.2799

53. .3887 54. 1.3634

55.  $\frac{5\pi}{6}$  56.  $\frac{2\pi}{3}$

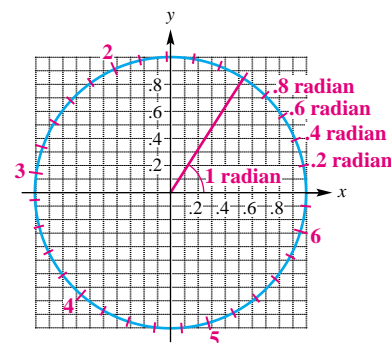
**Concept Check** The figure displays a unit circle and an angle of 1 radian. The tick marks on the circle are spaced at every two-tenths radian. Use the figure to estimate each value.

35.  $\cos .8$

36.  $\sin 4$

37. an angle whose cosine is  $-.65$

38. an angle whose sine is  $-.95$



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57.  $\frac{4\pi}{3}$  58.  $\frac{7\pi}{6}$  59.  $\frac{7\pi}{4}$   
 60.  $\frac{11\pi}{6}$  61.  $(-.8011, .5985)$   
 62.  $(-.9668, -.2555)$   
 63.  $(.4385, -.8987)$   
 64.  $(-.7259, .6878)$   
 65. I 66. IV 67. II 68. III  
 69.  $\frac{3\pi}{32}$  radian per sec  
 70.  $\frac{\pi}{25}$  radian per sec  
 71.  $\frac{6}{5}$  min 72. 9 min  
 73. .180311 radian per sec  
 74. 10.768 radians  
 75.  $\frac{9}{5}$  radians per sec  
 76. 6 radians per sec  
 77. 1.83333 radians per sec  
 78. 9.29755 cm per sec  
 79.  $18\pi$  cm 80.  $\frac{216\pi}{5}$  yd  
 81. 12 sec  
 82.  $\frac{3\pi}{32}$  radian per sec  
 83.  $\frac{\pi}{6}$  radian per hr  
 84.  $600\pi$  radians per min  
 85.  $\frac{7\pi}{30}$  cm per min

57.  $\left[\pi, \frac{3\pi}{2}\right]; \tan s = \sqrt{3}$   
 59.  $\left[\frac{3\pi}{2}, 2\pi\right]; \tan s = -1$

58.  $\left[\pi, \frac{3\pi}{2}\right]; \sin s = -\frac{1}{2}$   
 60.  $\left[\frac{3\pi}{2}, 2\pi\right]; \cos s = \frac{\sqrt{3}}{2}$

Suppose an arc of length  $s$  lies on the unit circle  $x^2 + y^2 = 1$ , starting at the point  $(1, 0)$  and terminating at the point  $(x, y)$ . (See Figure 10.) Use a calculator to find the approximate coordinates for  $(x, y)$ . (Hint:  $x = \cos s$  and  $y = \sin s$ .)

61.  $s = 2.5$  62.  $s = 3.4$  63.  $s = -7.4$  64.  $s = -3.9$

**Concept Check** For each value of  $s$ , use a calculator to find  $\sin s$  and  $\cos s$  and then use the results to decide in which quadrant an angle of  $s$  radians lies.

65.  $s = 51$  66.  $s = 49$  67.  $s = 65$  68.  $s = 79$

Use the formula  $\omega = \frac{\theta}{t}$  to find the value of the missing variable.

69.  $\theta = \frac{3\pi}{4}$  radians,  $t = 8$  sec 70.  $\theta = \frac{2\pi}{5}$  radians,  $t = 10$  sec  
 71.  $\theta = \frac{2\pi}{9}$  radian,  $\omega = \frac{5\pi}{27}$  radian per min  
 72.  $\theta = \frac{3\pi}{8}$  radians,  $\omega = \frac{\pi}{24}$  radian per min  
 73.  $\theta = 3.871142$  radians,  $t = 21.4693$  sec  
 74.  $\omega = .90674$  radian per min,  $t = 11.876$  min

Use the formula  $v = r\omega$  to find the value of the missing variable.

75.  $v = 9$  m per sec,  $r = 5$  m 76.  $v = 18$  ft per sec,  $r = 3$  ft  
 77.  $v = 107.692$  m per sec,  $r = 58.7413$  m  
 78.  $r = 24.93215$  cm,  $\omega = .372914$  radian per sec

The formula  $\omega = \frac{\theta}{t}$  can be rewritten as  $\theta = \omega t$ . Using  $\omega t$  for  $\theta$  changes  $s = r\theta$  to  $s = r\omega t$ . Use the formula  $s = r\omega t$  to find the value of the missing variable.

79.  $r = 6$  cm,  $\omega = \frac{\pi}{3}$  radians per sec,  $t = 9$  sec  
 80.  $r = 9$  yd,  $\omega = \frac{2\pi}{5}$  radians per sec,  $t = 12$  sec  
 81.  $s = 6\pi$  cm,  $r = 2$  cm,  $\omega = \frac{\pi}{4}$  radian per sec  
 82.  $s = \frac{3\pi}{4}$  km,  $r = 2$  km,  $t = 4$  sec

Find  $\omega$  for each of the following.

83. the hour hand of a clock  
 84. a line from the center to the edge of a CD revolving 300 times per min

Find  $v$  for each of the following.

85. the tip of the minute hand of a clock, if the hand is 7 cm long

86.  $1260\pi$  cm per min  
 87.  $1500\pi$  m per min  
 88.  $112,880\pi$  cm per min  
 89.  $2\pi$  sec  
 91. 15.5 mph 92. 24.62 hr  
 93. (a)  $\frac{2\pi}{365}$  radian  
 (b)  $\frac{\pi}{4380}$  radian per hr  
 (c) 66,700 mph  
 94. (a)  $2\pi$  radians per day;  
 $\frac{\pi}{12}$  radian per hr (b) 0  
 (c)  $12,800\pi$  km per day or about  
 $533\pi$  km per hr  
 (d) about 28,000 km per day or  
 about 1200 km per hr  
 95. (a) .24 radian per sec  
 (b) 3.11 cm per sec  
 96. larger pulley:  
 $\frac{25\pi}{18}$  radians per sec;  
 smaller pulley:  
 $\frac{125\pi}{48}$  radians per sec

86. a point on the tread of a tire of radius 18 cm, rotating 35 times per min  
 87. the tip of an airplane propeller 3 m long, rotating 500 times per min (*Hint:*  $r = 1.5$  m.)  
 88. a point on the edge of a gyroscope of radius 83 cm, rotating 680 times per min  
 89. **Concept Check** If a point moves around the circumference of the unit circle at the speed of 1 unit per sec, how long will it take for the point to move around the entire circle?  
 90. What is the difference between linear velocity and angular velocity?

Solve each problem. See Examples 5 and 6.

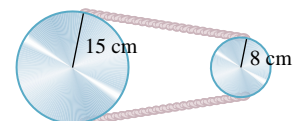
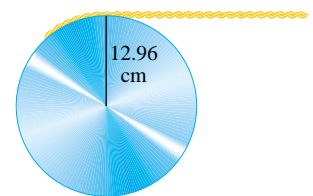
91. **Speed of a Bicycle** The tires of a bicycle have radius 13 in. and are turning at the rate of 200 revolutions per min. See the figure. How fast is the bicycle traveling in miles per hour? (*Hint:*  $5280$  ft = 1 mi.)



92. **Hours in a Martian Day** Mars rotates on its axis at the rate of about .2552 radian per hr. Approximately how many hours are in a Martian day? (*Source:* Wright, John W. (General Editor), *The Universal Almanac*, Andrews and McMeel, 1997.)



93. **Angular and Linear Speeds of Earth** Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.  
 (a) Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.  
 (b) Give the angular speed in radians per hour.  
 (c) Find the linear speed of Earth in miles per hour.  
 94. **Angular and Linear Speeds of Earth** Earth revolves on its axis once every 24 hr. Assuming that Earth's radius is 6400 km, find the following.  
 (a) angular speed of Earth in radians per day and radians per hour  
 (b) linear speed at the North Pole or South Pole  
 (c) linear speed at Quito, Ecuador, a city on the equator  
 (d) linear speed at Salem, Oregon (halfway from the equator to the North Pole)  
 95. **Speeds of a Pulley and a Belt** The pulley shown has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.  
 (a) Find the angular speed of the pulley in radians per second.  
 (b) Find the linear speed of the belt in centimeters per second.  
 96. **Angular Speed of Pulleys** The two pulleys in the figure have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in radians per second.





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97. 3.73 cm

98. about 29 sec

99. 523.6 radians per sec

100. 125 ft per sec

97. **Radius of a Spool of Thread** A thread is being pulled off a spool at the rate of 59.4 cm per sec. Find the radius of the spool if it makes 152 revolutions per min.

98. **Time to Move Along a Railroad Track** A railroad track is laid along the arc of a circle of radius 1800 ft. The circular part of the track subtends a central angle of  $40^\circ$ . How long (in seconds) will it take a point on the front of a train traveling 30 mph to go around this portion of the track?

99. **Angular Speed of a Motor Propeller** A 90-horsepower outboard motor at full throttle will rotate its propeller at 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

100. **Linear Speed of a Golf Club** The shoulder joint can rotate at about 25 radians per sec. If a golfer's arm is straight and the distance from the shoulder to the club head is 5 ft, estimate the linear speed of the club head from shoulder rotation. (Source: Cooper, J. and R. Glassow, *Kinesiology*, Second Edition, C.V. Mosby, 1968.)

## 6.3 Graphs of the Sine and Cosine Functions

Periodic Functions ■ Graph of the Sine Function ■ Graph of the Cosine Function ■ Graphing Techniques, Amplitude, and Period ■ Translations ■ Combinations of Translations ■ Determining a Trigonometric Model Using Curve Fitting



**Periodic Functions** Many things in daily life repeat with a predictable pattern: in warm areas electricity use goes up in summer and down in winter, the price of fresh fruit goes down in summer and up in winter, and attendance at amusement parks increases in spring and declines in autumn. Because the sine and cosine functions repeat their values in a regular pattern, they are **periodic functions**. Figure 19 shows a sine graph that represents a normal heartbeat.

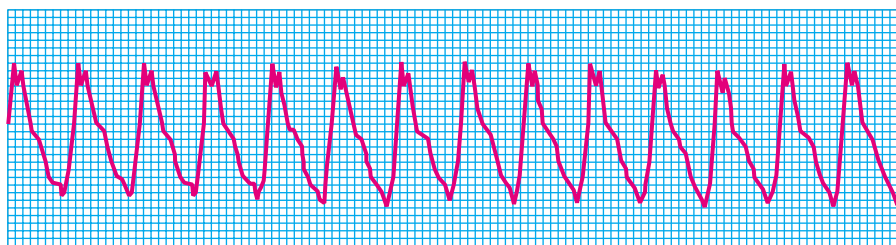


Figure 19

### Looking Ahead to Calculus

Periodic functions are used throughout calculus, so you will need to know their characteristics. One use of these functions is to describe the location of a point in the plane using *polar coordinates*, an alternative to rectangular coordinates. (See Chapter 8).

### Periodic Function

A **periodic function** is a function  $f$  such that

$$f(x) = f(x + np),$$

for every real number  $x$  in the domain of  $f$ , every integer  $n$ , and some positive real number  $p$ . The smallest possible positive value of  $p$  is the **period** of the function.

The circumference of the unit circle is  $2\pi$ , so the smallest value of  $p$  for which the sine and cosine functions repeat is  $2\pi$ . Therefore, the sine and cosine functions are periodic functions with period  $2\pi$ .



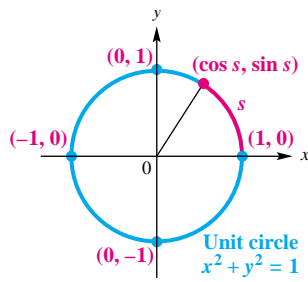
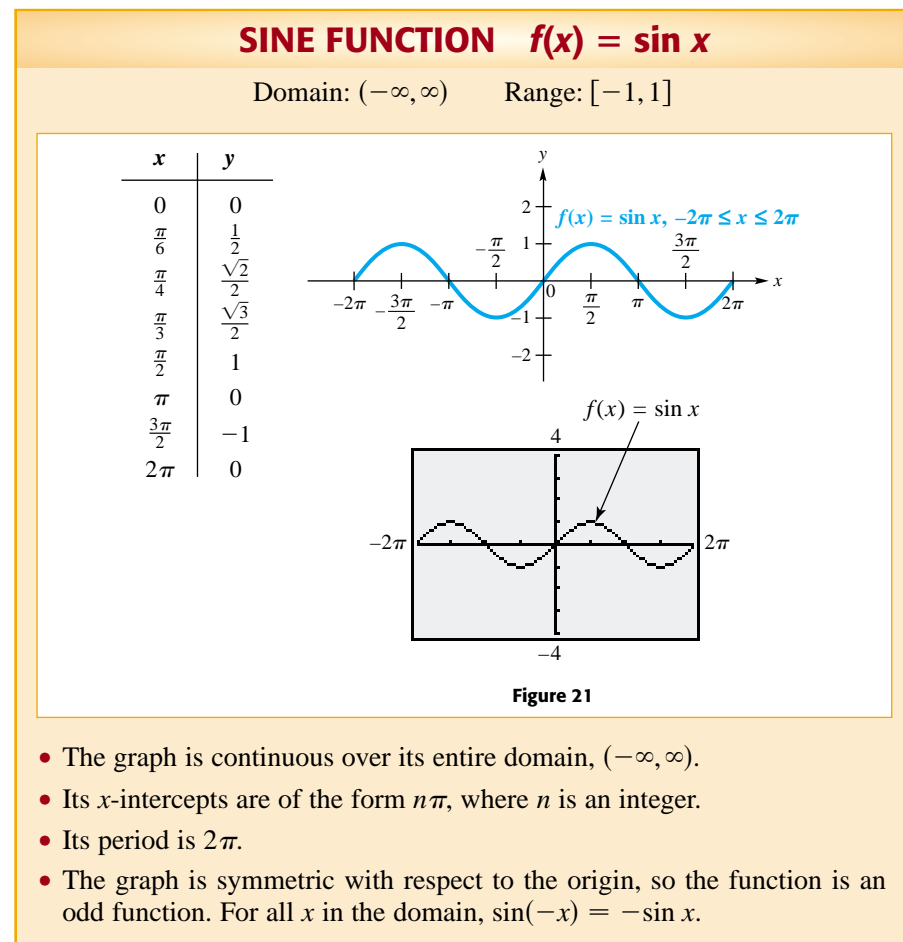


Figure 20

**Graph of the Sine Function** In Section 6.1 we saw that for a real number  $s$ , the point on the unit circle corresponding to  $s$  has coordinates  $(\cos s, \sin s)$ . See Figure 20. Trace along the circle to verify the results shown in the table.

As $s$ Increases from	$\sin s$	$\cos s$
0 to $\frac{\pi}{2}$	Increases from 0 to 1	Decreases from 1 to 0
$\frac{\pi}{2}$ to $\pi$	Decreases from 1 to 0	Decreases from 0 to $-1$
$\pi$ to $\frac{3\pi}{2}$	Decreases from 0 to $-1$	Increases from $-1$ to 0
$\frac{3\pi}{2}$ to $2\pi$	Increases from $-1$ to 0	Increases from 0 to 1

To avoid confusion when graphing the sine function, we use  $x$  rather than  $s$ ; this corresponds to the letters in the  $xy$ -coordinate system. Selecting key values of  $x$  and finding the corresponding values of  $\sin x$  leads to the table in Figure 21. To obtain the traditional graph in Figure 21, we plot the points from the table, use symmetry, and join them with a smooth curve. Since  $y = \sin x$  is periodic with period  $2\pi$  and has domain  $(-\infty, \infty)$ , the graph continues in the same pattern in both directions. This graph is called a **sine wave** or **sinusoid**.

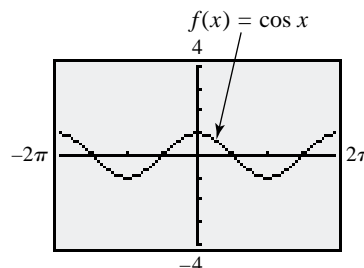
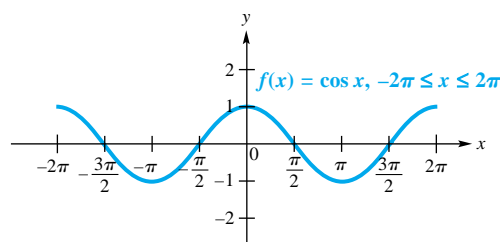


**Looking Ahead to Calculus**


The discussion of the derivative of a function in calculus shows that for the sine function, the slope of the tangent line at any point  $x$  is given by  $\cos x$ . For example, look at the graph of  $y = \sin x$  and notice that a tangent line at  $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$  will be horizontal and thus have slope 0. Now look at the graph of  $y = \cos x$  and see that for these values,  $\cos x = 0$ .

**Graph of the Cosine Function** We find the graph of  $y = \cos x$  in much the same way as the graph of  $y = \sin x$ . In the table of values shown with Figure 22 for  $y = \cos x$ , we use the same values for  $x$  as we did for the graph of  $y = \sin x$ . Notice that the graph of  $y = \cos x$  in Figure 22 has the same shape as the graph of  $y = \sin x$ . It is, in fact, the graph of the sine function shifted, or translated,  $\frac{\pi}{2}$  units to the left.

$x$	$y$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{\pi}{2}$	-1
$\pi$	0
$\frac{3\pi}{2}$	0
$2\pi$	1

**Figure 22**

- The graph is continuous over its entire domain,  $(-\infty, \infty)$ .
- Its  $x$ -intercepts are of the form  $(2n + 1)\frac{\pi}{2}$ , where  $n$  is an integer.
- Its period is  $2\pi$ .
- The graph is symmetric with respect to the  $y$ -axis, so the function is an even function. For all  $x$  in the domain,  $\cos(-x) = \cos x$ .

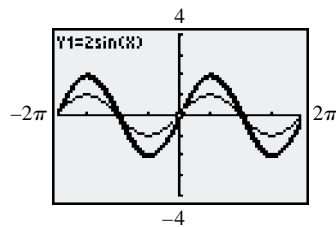
 Notice that the calculator graphs of  $f(x) = \sin x$  in Figure 21 and  $f(x) = \cos x$  in Figure 22 are graphed in the window  $[-2\pi, 2\pi]$  by  $[-4, 4]$ , with  $Xscl = \frac{\pi}{2}$  and  $Yscl = 1$ . This is called the **trig viewing window**. (Your model may use a different “standard” trigonometric viewing window. Consult your owner’s manual.) ■

**Graphing Techniques, Amplitude, and Period** The examples that follow show graphs that are “stretched” or “compressed” either vertically, horizontally, or both when compared with the graphs of  $y = \sin x$  or  $y = \cos x$ .

### EXAMPLE 1 Graphing $y = a \sin x$

Graph  $y = 2 \sin x$ , and compare to the graph of  $y = \sin x$ .

**Solution** For a given value of  $x$ , the value of  $y$  is twice as large as it would be for  $y = \sin x$ , as shown in the table of values. The only change in the graph is the range, which becomes  $[-2, 2]$ . See Figure 23, which includes a graph of  $y = \sin x$  for comparison.



The thick graph style represents the function in Example 1.

$x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin x$	0	1	0	-1	0
$2 \sin x$	0	2	0	-2	0

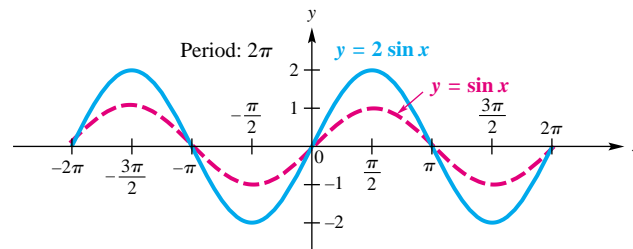


Figure 23

The **amplitude** of a periodic function is half the difference between the maximum and minimum values. Thus, for both the basic sine and cosine functions, the amplitude is

$$\frac{1}{2}[1 - (-1)] = \frac{1}{2}(2) = 1.$$

Generalizing from Example 1 gives the following.

### Amplitude

The graph of  $y = a \sin x$  or  $y = a \cos x$ , with  $a \neq 0$ , will have the same shape as the graph of  $y = \sin x$  or  $y = \cos x$ , respectively, except with range  $[-|a|, |a|]$ . The amplitude is  $|a|$ .

Now try Exercise 7.

**TEACHING TIP** Students may think that the period of  $y = \sin 2x$  is  $4\pi$ . Explain that a factor of 2 causes values of the argument to increase twice as fast, thereby shortening the length of a period.

No matter what the value of the amplitude, the periods of  $y = a \sin x$  and  $y = a \cos x$  are still  $2\pi$ . Consider  $y = \sin 2x$ . We can complete a table of values for the interval  $[0, 2\pi]$ .

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$\sin 2x$	0	1	0	-1	0	1	0	-1	0

Note that one complete cycle occurs in  $\pi$  units, not  $2\pi$  units. Therefore, the period here is  $\pi$ , which equals  $\frac{2\pi}{2}$ . Now consider  $y = \sin 4x$ . Look at the next table.

$x$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\pi$
$\sin 4x$	0	1	0	-1	0	1	0	-1	0

## 560 CHAPTER 6 The Circular Functions and Their Graphs

These values suggest that a complete cycle is achieved in  $\frac{\pi}{2}$  or  $\frac{2\pi}{4}$  units, which is reasonable since

$$\sin\left(4 \cdot \frac{\pi}{2}\right) = \sin 2\pi = 0.$$

In general, the graph of a function of the form  $y = \sin bx$  or  $y = \cos bx$ , for  $b > 0$ , will have a period different from  $2\pi$  when  $b \neq 1$ . To see why this is so, remember that the values of  $\sin bx$  or  $\cos bx$  will take on all possible values as  $bx$  ranges from 0 to  $2\pi$ . Therefore, to find the period of either of these functions, we must solve the three-part inequality

$$0 \leq bx \leq 2\pi \quad (\text{Section 1.7})$$

$$0 \leq x \leq \frac{2\pi}{b}. \quad \text{Divide by the positive number } b.$$

Thus, the period is  $\frac{2\pi}{b}$ . By dividing the interval  $[0, \frac{2\pi}{b}]$  into four equal parts, we obtain the values for which  $\sin bx$  or  $\cos bx$  is  $-1$ ,  $0$ , or  $1$ . These values will give minimum points,  $x$ -intercepts, and maximum points on the graph. Once these points are determined, we can sketch the graph by joining the points with a smooth sinusoidal curve. (If a function has  $b < 0$ , then the identities of the next chapter can be used to rewrite the function so that  $b > 0$ .)

**NOTE** One method to divide an interval into four equal parts is as follows.

**Step 1** Find the midpoint of the interval by adding the  $x$ -values of the endpoints and dividing by 2.

**Step 2** Find the midpoints of the two intervals found in Step 1, using the same procedure.

### EXAMPLE 2 Graphing $y = \sin bx$

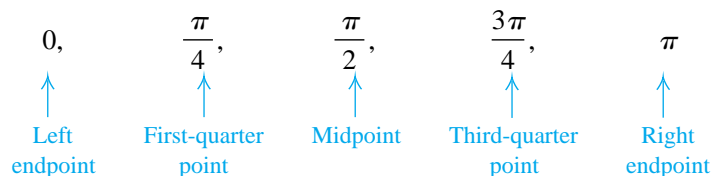
Graph  $y = \sin 2x$ , and compare to the graph of  $y = \sin x$ .

**Solution** In this function the coefficient of  $x$  is 2, so the period is  $\frac{2\pi}{2} = \pi$ . Therefore, the graph will complete one period over the interval  $[0, \pi]$ .

The endpoints are 0 and  $\pi$ , and the three middle points are

$$\frac{1}{4}(0 + \pi), \quad \frac{1}{2}(0 + \pi), \quad \text{and} \quad \frac{3}{4}(0 + \pi),$$

which give the following  $x$ -values.



We plot the points from the table of values given on page 559, and join them with a smooth sinusoidal curve. More of the graph can be sketched by repeating this cycle, as shown in Figure 24. The amplitude is not changed. The graph of  $y = \sin x$  is included for comparison.

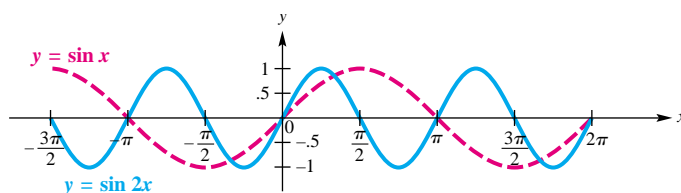


Figure 24

Now try Exercise 15.

Generalizing from Example 2 leads to the following result.

### Period

For  $b > 0$ , the graph of  $y = \sin bx$  will resemble that of  $y = \sin x$ , but with period  $\frac{2\pi}{b}$ . Also, the graph of  $y = \cos bx$  will resemble that of  $y = \cos x$ , but with period  $\frac{2\pi}{b}$ .

### EXAMPLE 3 Graphing $y = \cos bx$

Graph  $y = \cos \frac{2}{3}x$  over one period.

**Solution** The period is  $\frac{2\pi}{\frac{2}{3}} = 3\pi$ . We divide the interval  $[0, 3\pi]$  into four equal parts to get the following  $x$ -values that yield minimum points, maximum points, and  $x$ -intercepts.

$$0, \quad \frac{3\pi}{4}, \quad \frac{3\pi}{2}, \quad \frac{9\pi}{4}, \quad 3\pi$$

We use these values to obtain a table of key points for one period.

$x$	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	$3\pi$
$\frac{2}{3}x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\cos \frac{2}{3}x$	1	0	-1	0	1

The amplitude is 1 because the maximum value is 1, the minimum value is  $-1$ , and half of  $1 - (-1)$  is  $\frac{1}{2}(2) = 1$ . We plot these points and join them with a smooth curve. The graph is shown in Figure 25.

Now try Exercise 13.

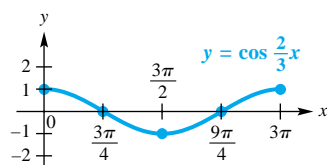
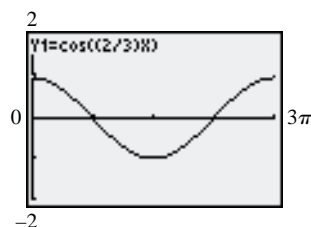


Figure 25



This screen shows a graph of the function in Example 3. By choosing  $X_{\text{scl}} = \frac{3\pi}{4}$ , the  $x$ -intercepts, maxima, and minima coincide with tick marks on the  $x$ -axis.

**NOTE** Look at the middle row of the table in Example 3. The method of dividing the interval  $[0, \frac{2\pi}{b}]$  into four equal parts will always give the values  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$  and  $2\pi$  for this row, resulting in values of  $-1, 0,$  or  $1$  for the circular function. These lead to key points on the graph, which can then be easily sketched.

The method used in Examples 1–3 is summarized as follows.

### Guidelines for Sketching Graphs of Sine and Cosine Functions

To graph  $y = a \sin bx$  or  $y = a \cos bx$ , with  $b > 0$ , follow these steps.

- Step 1** Find the period,  $\frac{2\pi}{b}$ . Start at 0 on the  $x$ -axis, and lay off a distance of  $\frac{2\pi}{b}$ .
- Step 2** Divide the interval into four equal parts. (See the Note preceding Example 2.)
- Step 3** Evaluate the function for each of the five  $x$ -values resulting from Step 2. The points will be maximum points, minimum points, and  $x$ -intercepts.
- Step 4** Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude  $|a|$ .
- Step 5** Draw the graph over additional periods, to the right and to the left, as needed.

The function in Example 4 has both amplitude and period affected by the values of  $a$  and  $b$ .

#### EXAMPLE 4 Graphing $y = a \sin bx$

Graph  $y = -2 \sin 3x$  over one period using the preceding guidelines.

##### Solution

**Step 1** For this function,  $b = 3$ , so the period is  $\frac{2\pi}{3}$ . The function will be graphed over the interval  $[0, \frac{2\pi}{3}]$ .

**Step 2** Divide the interval  $[0, \frac{2\pi}{3}]$  into four equal parts to get the  $x$ -values 0,  $\frac{\pi}{6}$ ,  $\frac{\pi}{3}$ ,  $\frac{\pi}{2}$ , and  $\frac{2\pi}{3}$ .

**Step 3** Make a table of values determined by the  $x$ -values from Step 2.

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin 3x$	0	1	0	-1	0
$-2 \sin 3x$	0	-2	0	2	0

**Step 4** Plot the points  $(0, 0)$ ,  $(\frac{\pi}{6}, -2)$ ,  $(\frac{\pi}{3}, 0)$ ,  $(\frac{\pi}{2}, 2)$ , and  $(\frac{2\pi}{3}, 0)$ , and join them with a sinusoidal curve with amplitude 2. See Figure 26.

**Step 5** The graph can be extended by repeating the cycle.

Notice that when  $a$  is negative, the graph of  $y = a \sin bx$  is the reflection across the  $x$ -axis of the graph of  $y = |a| \sin bx$ .

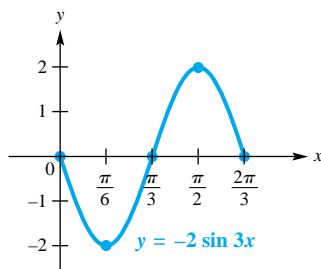


Figure 26

Now try Exercise 19.

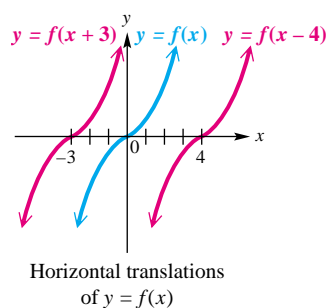


Figure 27

**Translations** In general, the graph of the function defined by  $y = f(x - d)$  is translated *horizontally* when compared to the graph of  $y = f(x)$ . The translation is  $d$  units to the right if  $d > 0$  and  $|d|$  units to the left if  $d < 0$ . See Figure 27. With trigonometric functions, a horizontal translation is called a **phase shift**. In the function  $y = f(x - d)$ , the expression  $x - d$  is called the **argument**.

In Example 5, we give two methods that can be used to sketch the graph of a circular function involving a phase shift.

**EXAMPLE 5** Graphing  $y = \sin(x - \frac{\pi}{3})$

Graph  $y = \sin\left(x - \frac{\pi}{3}\right)$ .

**Solution Method 1** For the argument  $x - \frac{\pi}{3}$  to result in all possible values throughout one period, it must take on all values between 0 and  $2\pi$ , inclusive. Therefore, to find an interval of one period, we solve the three-part inequality

$$\begin{aligned} 0 &\leq x - \frac{\pi}{3} \leq 2\pi \\ \frac{\pi}{3} &\leq x \leq \frac{7\pi}{3}. \quad \text{Add } \frac{\pi}{3} \text{ to each part.} \end{aligned}$$

Divide the interval  $[\frac{\pi}{3}, \frac{7\pi}{3}]$  into four equal parts to get the following  $x$ -values.

$$\frac{\pi}{3}, \quad \frac{5\pi}{6}, \quad \frac{4\pi}{3}, \quad \frac{11\pi}{6}, \quad \frac{7\pi}{3}$$

A table of values using these  $x$ -values follows.

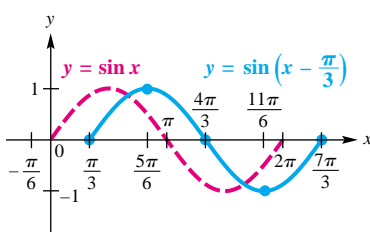


Figure 28

$x$	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$
$x - \frac{\pi}{3}$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin(x - \frac{\pi}{3})$	0	1	0	-1	0

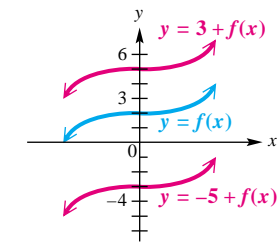
We join the corresponding points to get the graph shown in Figure 28. The period is  $2\pi$ , and the amplitude is 1.

**Method 2** We can also graph  $y = \sin(x - \frac{\pi}{3})$  using a horizontal translation. The argument  $x - \frac{\pi}{3}$  indicates that the graph will be translated  $\frac{\pi}{3}$  units to the *right* (the phase shift) as compared to the graph of  $y = \sin x$ . In Figure 28 we show the graph of  $y = \sin x$  as a dashed curve, and the graph of  $y = \sin(x - \frac{\pi}{3})$  as a solid curve. Therefore, to graph a function using this method, first graph the basic circular function, and then graph the desired function by using the appropriate translation.

The graph can be extended through additional periods by repeating this portion of the graph over and over, as necessary.

Now try Exercise 37.

The graph of a function of the form  $y = c + f(x)$  is translated *vertically* as compared with the graph of  $y = f(x)$ . See Figure 29. The translation is  $c$  units up if  $c > 0$  and  $|c|$  units down if  $c < 0$ .



Vertical translations of  $y = f(x)$

Figure 29

**EXAMPLE 6** Graphing  $y = c + a \cos bx$

Graph  $y = 3 - 2 \cos 3x$ .

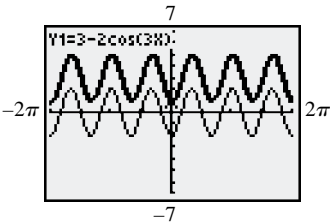
**Solution** The values of  $y$  will be 3 greater than the corresponding values of  $y$  in  $y = -2 \cos 3x$ . This means that the graph of  $y = 3 - 2 \cos 3x$  is the same as the graph of  $y = -2 \cos 3x$ , vertically translated 3 units up. Since the period of  $y = -2 \cos 3x$  is  $\frac{2\pi}{3}$ , the key points have  $x$ -values

$$0, \quad \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}.$$

Use these  $x$ -values to make a table of points.

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$\cos 3x$	1	0	-1	0	1
$2 \cos 3x$	2	0	-2	0	2
$3 - 2 \cos 3x$	1	3	5	3	1

The key points are shown on the graph in Figure 30, along with more of the graph, sketched using the fact that the function is periodic.



The function in Example 6 is shown using the thick graph style. Notice also the thin graph style for  $y = -2 \cos 3x$ .

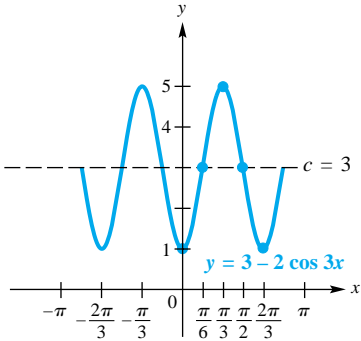


Figure 30

Now try Exercise 45.



**Combinations of Translations** A function of the form

$$y = c + a \sin b(x - d) \quad \text{or} \quad y = c + a \cos b(x - d), \quad b > 0,$$

can be graphed according to the following guidelines.

**TEACHING TIP** Give several examples of sinusoidal graphs, and ask students to determine their corresponding equations. Point out that any given graph may have several different corresponding equations, all of which are valid.

Many computer graphing utilities have animation features that allow you to incrementally change the values of  $a$ ,  $b$ ,  $c$ , and  $d$  in the sinusoidal graphs of

$$y = c + a \sin b(x - d)$$

$$\text{and } y = c + a \cos b(x - d).$$

This provides an excellent visual image as to how these parameters affect sinusoidal graphs.

**Further Guidelines for Sketching Graphs of Sine and Cosine Functions**

**Method 1** Follow these steps.

**Step 1** Find an interval whose length is one period  $\frac{2\pi}{b}$  by solving the three-part inequality  $0 \leq b(x - d) \leq 2\pi$ .

**Step 2** Divide the interval into four equal parts.

**Step 3** Evaluate the function for each of the five  $x$ -values resulting from Step 2. The points will be maximum points, minimum points, and points that intersect the line  $y = c$  (“middle” points of the wave).

**Step 4** Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude  $|a|$ .

**Step 5** Draw the graph over additional periods, to the right and to the left, as needed.

**Method 2** First graph the basic circular function. The amplitude of the function is  $|a|$ , and the period is  $\frac{2\pi}{b}$ . Then use translations to graph the desired function. The vertical translation is  $c$  units up if  $c > 0$  and  $|c|$  units down if  $c < 0$ . The horizontal translation (phase shift) is  $d$  units to the right if  $d > 0$  and  $|d|$  units to the left if  $d < 0$ .

**EXAMPLE 7** Graphing  $y = c + a \sin b(x - d)$ 

Graph  $y = -1 + 2 \sin(4x + \pi)$ .

**Solution** We use Method 1. First write the expression in the form  $c + a \sin b(x - d)$  by rewriting  $4x + \pi$  as  $4\left(x + \frac{\pi}{4}\right)$ :

$$y = -1 + 2 \sin \left[ 4 \left( x + \frac{\pi}{4} \right) \right]. \quad \text{Rewrite } 4x + \pi \text{ as } 4\left(x + \frac{\pi}{4}\right).$$

**Step 1** Find an interval whose length is one period.

$$0 \leq 4 \left( x + \frac{\pi}{4} \right) \leq 2\pi$$

$$0 \leq x + \frac{\pi}{4} \leq \frac{\pi}{2} \quad \text{Divide by 4.}$$

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \quad \text{Subtract } \frac{\pi}{4}.$$

**Step 2** Divide the interval  $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$  into four equal parts to get the  $x$ -values

$$-\frac{\pi}{4}, \quad -\frac{\pi}{8}, \quad 0, \quad \frac{\pi}{8}, \quad \frac{\pi}{4}.$$

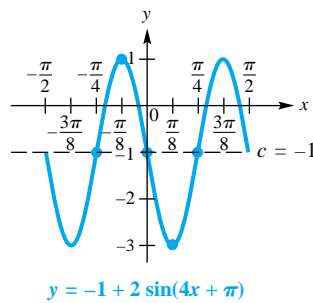


Figure 31

Step 3 Make a table of values.

$x$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	$0$	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$x + \frac{\pi}{4}$	$0$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$4(x + \frac{\pi}{4})$	$0$	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin 4(x + \frac{\pi}{4})$	$0$	$1$	$0$	$-1$	$0$
$2 \sin 4(x + \frac{\pi}{4})$	$0$	$2$	$0$	$-2$	$0$
$-1 + 2 \sin(4x + \pi)$	$-1$	$1$	$-1$	$-3$	$-1$

Steps 4 and 5 Plot the points found in the table and join them with a sinusoidal curve. Figure 31 shows the graph, extended to the right and left to include two full periods.

Now try Exercise 49.

**Determining a Trigonometric Model Using Curve Fitting** A sinusoidal function is often a good approximation of a set of real data points.



### EXAMPLE 8 Modeling Temperature with a Sine Function

The maximum average monthly temperature in New Orleans is  $82^{\circ}\text{F}$  and the minimum is  $54^{\circ}\text{F}$ . The table shows the average monthly temperatures. The scatter diagram for a 2-year interval in Figure 32 strongly suggests that the temperatures can be modeled with a sine curve.

Month	$^{\circ}\text{F}$	Month	$^{\circ}\text{F}$
Jan	54	July	82
Feb	55	Aug	81
Mar	61	Sept	77
Apr	69	Oct	71
May	73	Nov	59
June	79	Dec	55

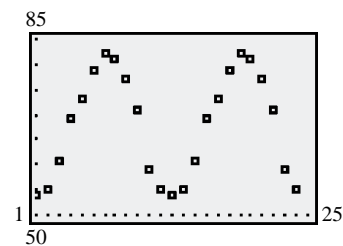


Figure 32

Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology*, 4th Edition, Charles E. Merrill Publishing Co., 1983.

- (a) Using only the maximum and minimum temperatures, determine a function of the form  $f(x) = a \sin[b(x - d)] + c$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants, that models the average monthly temperature in New Orleans. Let  $x$  represent the month, with January corresponding to  $x = 1$ .

- (b) On the same coordinate axes, graph  $f$  for a two-year period together with the actual data values found in the table.
- (c) Use the *sine regression* feature of a graphing calculator to determine a second model for these data.

### Solution

- (a) We use the maximum and minimum average monthly temperatures to find the amplitude  $a$ .

$$a = \frac{82 - 54}{2} = 14$$

The average of the maximum and minimum temperatures is a good choice for  $c$ . The average is

$$\frac{82 + 54}{2} = 68.$$

Since the coldest month is January, when  $x = 1$ , and the hottest month is July, when  $x = 7$ , we should choose  $d$  to be about 4. We experiment with values just greater than 4 to find  $d$ . Trial and error using a calculator leads to  $d = 4.2$ . Since temperatures repeat every 12 months,  $b$  is  $\frac{2\pi}{12} = \frac{\pi}{6}$ . Thus,

$$f(x) = a \sin[b(x - d)] + c = 14 \sin\left[\frac{\pi}{6}(x - 4.2)\right] + 68.$$

- (b) Figure 33 shows the data points and the graph of  $y = 14 \sin \frac{\pi}{6}x + 68$  for comparison. The horizontal translation of the model is fairly obvious here.

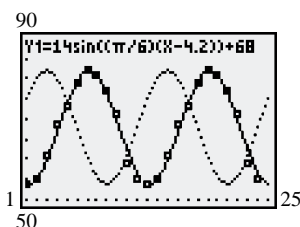
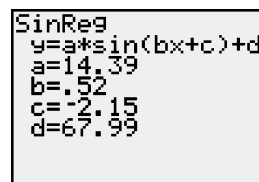
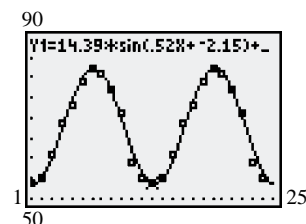


Figure 33



Values are rounded to the nearest hundredth.

(a)



(b)

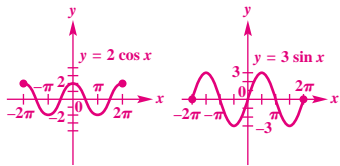
Figure 34

- (c) We used the given data for a two-year period to produce the model described in Figure 34(a). Figure 34(b) shows its graph along with the data points.

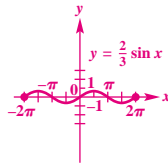
Now try Exercise 73.

## 6.3 Exercises

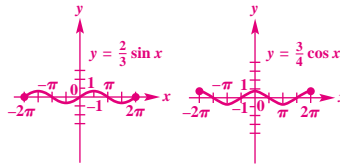
1. D 2. C 3. A 4. B  
5. 2 6. 3



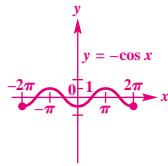
7.  $\frac{2}{3}$



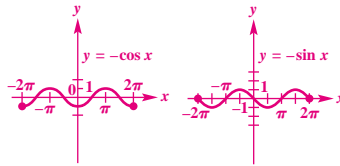
8.  $\frac{3}{4}$



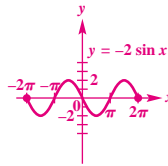
9. 1



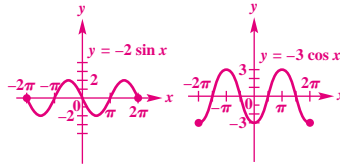
10. 1



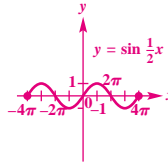
11. 2



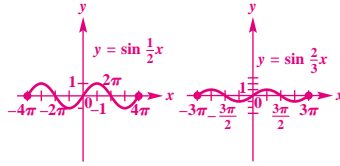
12. 3



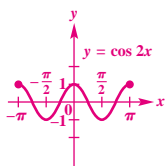
13.  $4\pi$ ; 1



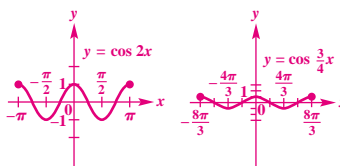
14.  $3\pi$ ; 1



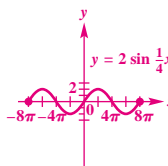
15.  $\pi$ ; 1



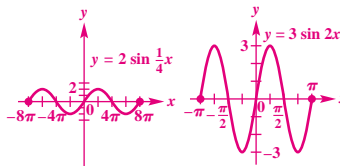
16.  $\frac{8\pi}{3}$ ; 1



17.  $8\pi$ ; 2



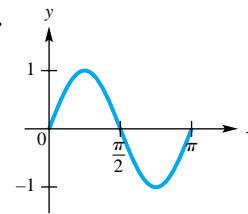
18.  $\pi$ ; 3



**Concept Check** In Exercises 1–4, match each function with its graph.

1.  $y = -\sin x$

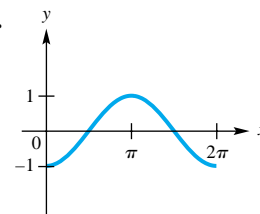
A.



2.  $y = -\cos x$

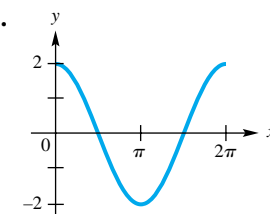
3.  $y = \sin 2x$

C.

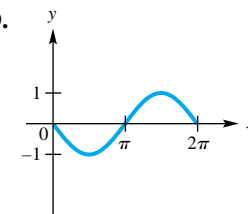


4.  $y = 2 \cos x$

B.



D.



Graph each function over the interval  $[-2\pi, 2\pi]$ . Give the amplitude. See Example 1.

5.  $y = 2 \cos x$

6.  $y = 3 \sin x$

7.  $y = \frac{2}{3} \sin x$

8.  $y = \frac{3}{4} \cos x$

9.  $y = -\cos x$

10.  $y = -\sin x$

11.  $y = -2 \sin x$

12.  $y = -3 \cos x$

Graph each function over a two-period interval. Give the period and amplitude. See Examples 2–4.

13.  $y = \sin \frac{1}{2}x$

14.  $y = \sin \frac{2}{3}x$

15.  $y = \cos 2x$

16.  $y = \cos \frac{3}{4}x$

17.  $y = 2 \sin \frac{1}{4}x$

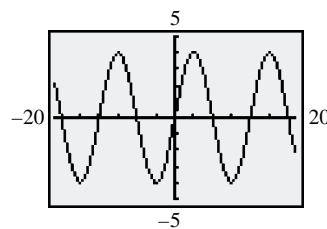
18.  $y = 3 \sin 2x$

19.  $y = -2 \cos 3x$

20.  $y = -5 \cos 2x$

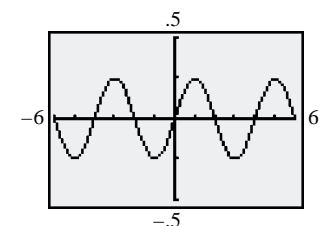
**Concept Check** In Exercises 21 and 22, give the equation of a sine function having the given graph.

21.



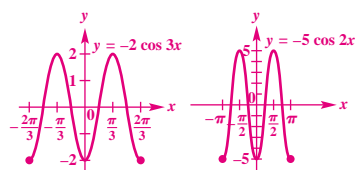
```
WINDOW
Xmin=-20
Xmax=20
Xscl=3.1415926...
Ymin=-5
Ymax=5
Yscl=1
Xres=1
```

22.



```
WINDOW
Xmin=-6
Xmax=6
Xscl=1
Ymin=-.5
Ymax=.5
Yscl=.25
Xres=1
```

19.  $\frac{2\pi}{3}; 2$       20.  $\pi; 5$



There are other correct answers in Exercises 21 and 22.

21.  $y = 4 \sin \frac{1}{2}x$

22.  $y = \frac{1}{4} \sin \frac{\pi}{2}x$

23. B    24. D    25. C    26. A

27. D    28. C    29. B    30. A

31. 2;  $2\pi$ ; none;  $\pi$  to the right

32.  $\frac{2}{3}$ ;  $2\pi$ ; none;  $\frac{\pi}{2}$  to the left

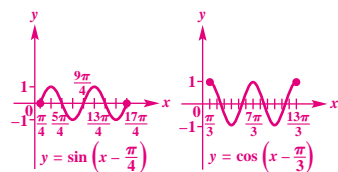
33. 4;  $4\pi$ ; none;  $\pi$  to the left

34.  $\frac{1}{2}$ ;  $4\pi$ ; none;  $2\pi$  to the left

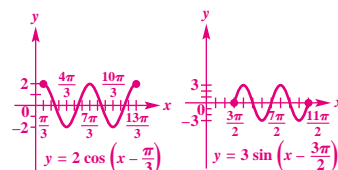
35. 1;  $\frac{2\pi}{3}$ ; up 2;  $\frac{\pi}{15}$  to the right

36.  $\frac{1}{2}$ ;  $\pi$ ; down 1;  $\frac{3\pi}{2}$  to the right

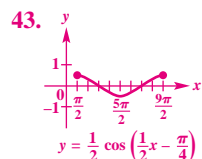
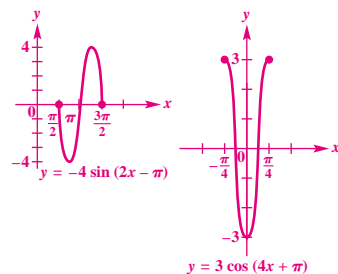
37.      38.



39.      40.



41.      42.



**Concept Check** Match each function in Column I with the appropriate description in Column II.

I

23.  $y = 3 \sin(2x - 4)$

24.  $y = 2 \sin(3x - 4)$

25.  $y = 4 \sin(3x - 2)$

26.  $y = 2 \sin(4x - 3)$

II

A. amplitude = 2, period =  $\frac{\pi}{2}$ , phase shift =  $\frac{3}{4}$

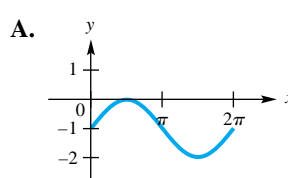
B. amplitude = 3, period =  $\pi$ , phase shift = 2

C. amplitude = 4, period =  $\frac{2\pi}{3}$ , phase shift =  $\frac{2}{3}$

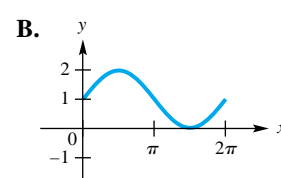
D. amplitude = 2, period =  $\frac{2\pi}{3}$ , phase shift =  $\frac{4}{3}$

**Concept Check** Match each function with its graph.

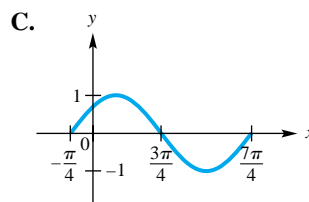
27.  $y = \sin\left(x - \frac{\pi}{4}\right)$



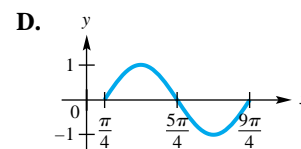
28.  $y = \sin\left(x + \frac{\pi}{4}\right)$



29.  $y = 1 + \sin x$



30.  $y = -1 + \sin x$



Find the amplitude, the period, any vertical translation, and any phase shift of the graph of each function. See Examples 5–7.

31.  $y = 2 \sin(x - \pi)$

32.  $y = \frac{2}{3} \sin\left(x + \frac{\pi}{2}\right)$

33.  $y = 4 \cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$

34.  $y = \frac{1}{2} \sin\left(\frac{x}{2} + \pi\right)$

35.  $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

36.  $y = -1 + \frac{1}{2} \cos(2x - 3\pi)$

Graph each function over a two-period interval. See Example 5.

37.  $y = \sin\left(x - \frac{\pi}{4}\right)$

38.  $y = \cos\left(x - \frac{\pi}{3}\right)$

39.  $y = 2 \cos\left(x - \frac{\pi}{3}\right)$

40.  $y = 3 \sin\left(x - \frac{3\pi}{2}\right)$

Graph each function over a one-period interval.

41.  $y = -4 \sin(2x - \pi)$

42.  $y = 3 \cos(4x + \pi)$

43.  $y = \frac{1}{2} \cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

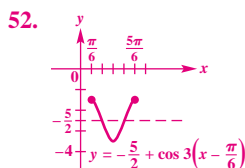
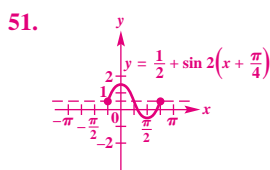
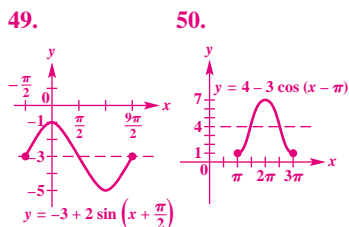
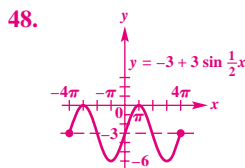
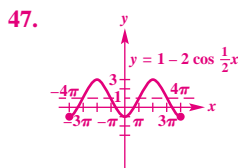
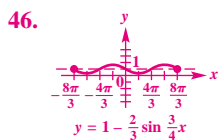
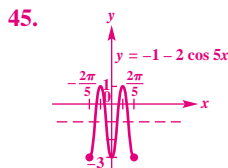
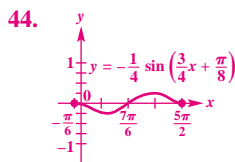
44.  $y = -\frac{1}{4} \sin\left(\frac{3}{4}x + \frac{\pi}{8}\right)$

Graph each function over a two-period interval. See Example 6.

45.  $y = -1 - 2 \cos 5x$

46.  $y = 1 - \frac{2}{3} \sin \frac{3}{4}x$

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There are other correct answers in Exercises 53 and 54.

53.  $y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$

54.  $y = \pi \sin \pi(x - .5)$

55. (a)  $80^\circ$ ;  $50^\circ$  (b)  $15^\circ$

(c) about 35,000 yr (d) downward

47.  $y = 1 - 2 \cos \frac{1}{2}x$

48.  $y = -3 + 3 \sin \frac{1}{2}x$

Graph each function over a one-period interval. See Example 7.

49.  $y = -3 + 2 \sin\left(x + \frac{\pi}{2}\right)$

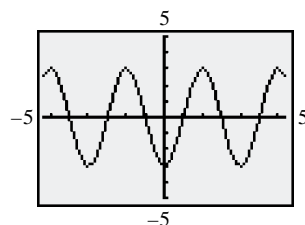
50.  $y = 4 - 3 \cos(x - \pi)$

51.  $y = \frac{1}{2} + \sin 2\left(x + \frac{\pi}{4}\right)$

52.  $y = -\frac{5}{2} + \cos 3\left(x - \frac{\pi}{6}\right)$

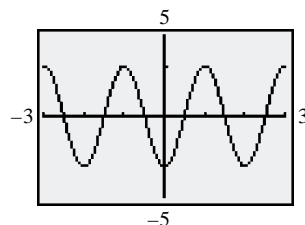
**Concept Check** In Exercises 53 and 54, find the equation of a sine function having the given graph.

53. (Note:  $X_{\text{scl}} = \frac{\pi}{4}$ .)



```
WINDOW
Xmin=-5
Xmax=5
Xscl=.78539816...
Ymin=-5
Ymax=5
Yscl=1
Xres=1
```

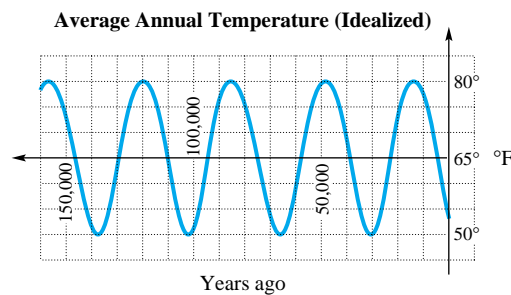
54. (Note:  $Y_{\text{scl}} = \pi$ .)



```
WINDOW
Xmin=-3
Xmax=3
Xscl=1
Ymin=-5
Ymax=5
Yscl=3.1415926...
Xres=1
```

(Modeling) Solve each problem.

55. **Average Annual Temperature** Scientists believe that the average annual temperature in a given location is periodic. The average temperature at a given place during a given season fluctuates as time goes on, from colder to warmer, and back to colder. The graph shows an idealized description of the temperature (in  $^\circ\text{F}$ ) for the last few thousand years of a location at the same latitude as Anchorage, Alaska.



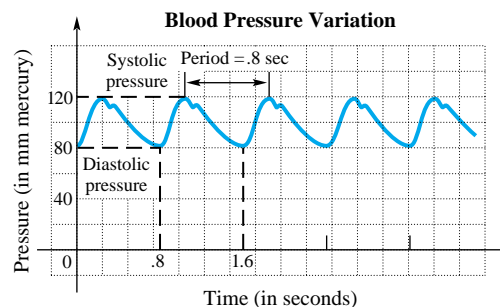
- Find the highest and lowest temperatures recorded.
- Use these two numbers to find the amplitude.
- Find the period of the function.
- What is the trend of the temperature now?

56. (a) 20 (b) 75

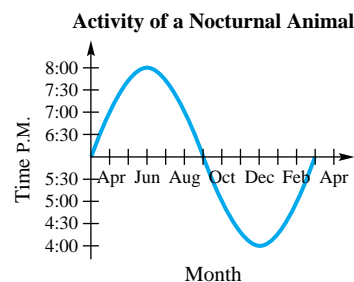
57. (a) about 2 hr (b) 1 yr

58. (a)  $y = \frac{1}{3} \sin \frac{4\pi t}{3}$  (b)  $\frac{3}{2}$  sec

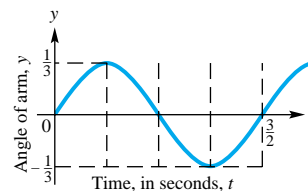
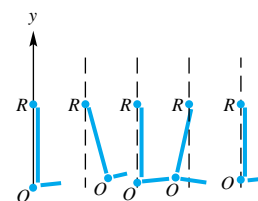
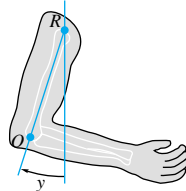
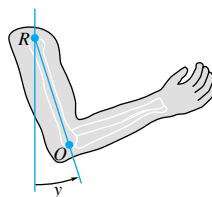
56. **Blood Pressure Variation** The graph gives the variation in blood pressure for a typical person. Systolic and diastolic pressures are the upper and lower limits of the periodic changes in pressure that produce the pulse. The length of time between peaks is called the period of the pulse.



- (a) Find the amplitude of the graph.  
 (b) Find the pulse rate (the number of pulse beats in 1 min) for this person.
57. **Activity of a Nocturnal Animal** Many of the activities of living organisms are periodic. For example, the graph below shows the time that a certain nocturnal animal begins its evening activity.



- (a) Find the amplitude of this graph. (b) Find the period.
58. **Position of a Moving Arm** The figure shows schematic diagrams of a rhythmically moving arm. The upper arm  $RO$  rotates back and forth about the point  $R$ ; the position of the arm is measured by the angle  $y$  between the actual position and the downward vertical position. (Source: De Sapio, Rodolfo, *Calculus for the Life Sciences*. Copyright © 1978 by W. H. Freeman and Company. Reprinted by permission.)

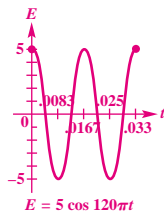


This graph shows the relationship between angle  $y$  and time  $t$  in seconds.

- (a) Find an equation of the form  $y = a \sin kt$  for the graph shown.  
 (b) How long does it take for a complete movement of the arm?

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59. 24 hr

60. approximately  $\frac{2.6 - .2}{2} = 1.2$ 61. approximately 6:00 P.M.;  
approximately .2 ft62. approximately 7:19 P.M.;  
approximately 0 ft63. approximately 2:00 A.M.;  
approximately 2.6 ft64. approximately 3:18 A.M.;  
approximately 2.4 ft65. 1;  $240^\circ$  or  $\frac{4\pi}{3}$ 66. 1;  $120^\circ$  or  $\frac{2\pi}{3}$ 67. (a) 5;  $\frac{1}{60}$  (b) 60(c) 5; 1.545; -4.045; -4.045;  
1.545 (d)

**Tides for Kahului Harbor** The chart shows the tides for Kahului Harbor (on the island of Maui, Hawaii). To identify high and low tides and times for other Maui areas, the following adjustments must be made.

Hana: High, +40 min, +.1 ft;

Low, +18 min, -.2 ft

Maalaea: High, +1:52, -.1 ft;

Low, +1:19, -.2 ft

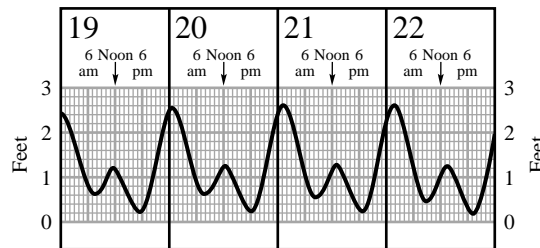
Makena: High, +1:21, -.5 ft;

Low, +1:09, -.2 ft

Lahaina: High, +1:18, -.2 ft;

Low, +1:01, -.1 ft

## JANUARY



Source: Maui News. Original chart prepared by Edward K. Noda and Associates.

Use the graph to work Exercises 59–64.

59. The graph is an example of a periodic function. What is the period (in hours)?

60. What is the amplitude?

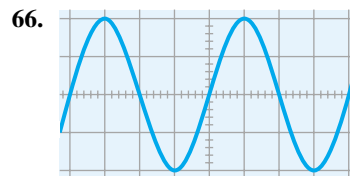
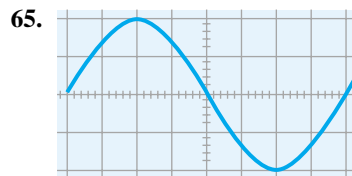
61. At what time on January 20 was low tide at Kahului? What was the height?

62. Repeat Exercise 61 for Maalaea.

63. At what time on January 22 was high tide at Kahului? What was the height?

64. Repeat Exercise 63 for Lahaina.

**Musical Sound Waves** Pure sounds produce single sine waves on an oscilloscope. Find the amplitude and period of each sine wave graph in Exercises 65 and 66. On the vertical scale, each square represents .5; on the horizontal scale, each square represents  $30^\circ$  or  $\frac{\pi}{6}$ .



(Modeling) Solve each problem.

67. **Voltage of an Electrical Circuit** The voltage  $E$  in an electrical circuit is modeled by

$$E = 5 \cos 120\pi t,$$

where  $t$  is time measured in seconds.

(a) Find the amplitude and the period.

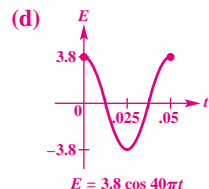
(b) How many cycles are completed in 1 sec? (The number of cycles (periods) completed in 1 sec is the frequency of the function.)

(c) Find  $E$  when  $t = 0, .03, .06, .09, .12$ .(d) Graph  $E$  for  $0 \leq t \leq \frac{1}{30}$ .



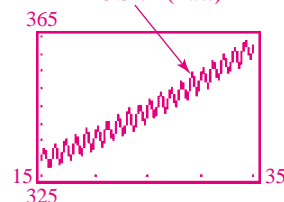
68. (a)  $3.8$ ;  $\frac{1}{20}$  (b) 20

(c)  $-3.074$ ;  $1.174$ ;  $-3.074$ ;  $-3.074$ ;  $1.174$



69. (a)

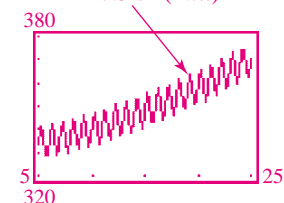
$$L(x) = .022x^2 + .55x + 316 + 3.5 \sin(2\pi x)$$



(b) maximums:  $x = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \dots$ ; minimums:  $x = \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \dots$

70. (a)

$$C(x) = .04x^2 + .6x + 330 + 7.5 \sin(2\pi x)$$



(c)  $C(x) = .04(x - 1970)^2 + .6(x - 1970) + 330 + 7.5 \sin[2\pi(x - 1970)]$

71. (a)  $1^\circ$  (b)  $19^\circ$  (c)  $53^\circ$  (d)  $58^\circ$  (e)  $48^\circ$  (f)  $12^\circ$

68. **Voltage of an Electrical Circuit** For another electrical circuit, the voltage  $E$  is modeled by

$$E = 3.8 \cos 40\pi t,$$

where  $t$  is time measured in seconds.

- (a) Find the amplitude and the period.  
 (b) Find the frequency. See Exercise 67(b).  
 (c) Find  $E$  when  $t = .02, .04, .08, .12, .14$ .  
 (d) Graph one period of  $E$ .

69. **Atmospheric Carbon Dioxide** At Mauna Loa, Hawaii, atmospheric carbon dioxide levels in parts per million (ppm) have been measured regularly since 1958. The function defined by

$$L(x) = .022x^2 + .55x + 316 + 3.5 \sin(2\pi x)$$

can be used to model these levels, where  $x$  is in years and  $x = 0$  corresponds to 1960. (Source: Nilsson, A., *Greenhouse Earth*, John Wiley & Sons, 1992.)



(a) Graph  $L$  in the window  $[15, 35]$  by  $[325, 365]$ .

(b) When do the seasonal maximum and minimum carbon dioxide levels occur?

(c)  $L$  is the sum of a quadratic function and a sine function. What is the significance of each of these functions? Discuss what physical phenomena may be responsible for each function.

70. **Atmospheric Carbon Dioxide** Refer to Exercise 69. The carbon dioxide content in the atmosphere at Barrow, Alaska, in parts per million (ppm) can be modeled using the function defined by

$$C(x) = .04x^2 + .6x + 330 + 7.5 \sin(2\pi x),$$

where  $x = 0$  corresponds to 1970. (Source: Zeilik, M. and S. Gregory, *Introductory Astronomy and Astrophysics*, Brooks/Cole, 1998.)

(a) Graph  $C$  in the window  $[5, 25]$  by  $[320, 380]$ .

(b) Discuss possible reasons why the amplitude of the oscillations in the graph of  $C$  is larger than the amplitude of the oscillations in the graph of  $L$  in Exercise 69, which models Hawaii.

(c) Define a new function  $C$  that is valid if  $x$  represents the actual year, where  $1970 \leq x \leq 1995$ .

71. **Temperature in Fairbanks** The temperature in Fairbanks is modeled by

$$T(x) = 37 \sin \left[ \frac{2\pi}{365} (x - 101) \right] + 25,$$

where  $T(x)$  is the temperature in degrees Fahrenheit on day  $x$ , with  $x = 1$  corresponding to January 1 and  $x = 365$  corresponding to December 31. Use a calculator to estimate the temperature on the following days. (Source: Lando, B. and C. Lando, "Is the Graph of Temperature Variation a Sine Curve?", *The Mathematics Teacher*, 70, September 1977.)

- (a) March 1 (day 60) (b) April 1 (day 91) (c) Day 150  
 (d) June 15 (e) September 1 (f) October 31

72. **Fluctuation in the Solar Constant** The solar constant  $S$  is the amount of energy per unit area that reaches Earth's atmosphere from the sun. It is equal to 1367 watts

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72. (a) 1.998 watts per m<sup>2</sup>

(b) -46.461 watts per m<sup>2</sup>

(c) 46.478 watts per m<sup>2</sup>

(d) Answers may vary. A possible answer is  $N = 82.5$ . (Since  $N$  represents a day number, which should be a natural number, we might interpret day 82.5 as noon on the 82nd day.)

Answer graphs for Exercises 73(a), (b), and (e) are included on page A-38 of the answer section at the back of the text.

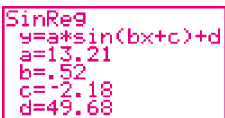
73. (a) yes (b) It represents the average yearly temperature.

(c) 14; 12; 4.2

(d)  $f(x) =$

$$14 \sin \left[ \frac{\pi}{6} (x - 4.2) \right] + 50$$

(e) The function gives an excellent model for the given data.

(f) 

TI-83 Plus fixed to the nearest hundredth.

74. (a) 70.4°

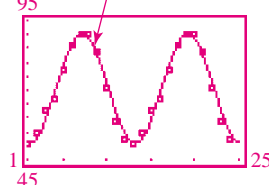
(b) See the graph in part (d).

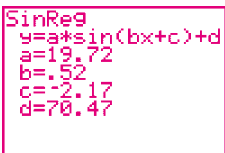
(c)  $f(x) =$

$$19.5 \cos \left[ \frac{\pi}{6} (x - 7.2) \right] + 70.5$$

(d) The function gives an excellent model for the data.

$$f(x) = 19.5 \cos \left[ \frac{\pi}{6} (x - 7.2) \right] + 70.5$$



(e) 

TI-83 Plus fixed to the nearest hundredth.

per square meter but varies slightly throughout the seasons. This fluctuation  $\Delta S$  in  $S$  can be calculated using the formula

$$\Delta S = .034S \sin \left[ \frac{2\pi(82.5 - N)}{365.25} \right].$$

In this formula,  $N$  is the day number covering a four-year period, where  $N = 1$  corresponds to January 1 of a leap year and  $N = 1461$  corresponds to December 31 of the fourth year. (Source: Winter, C., R. Sizmann, and Vant-Hunt (Editors), *Solar Power Plants*, Springer-Verlag, 1991.)

(a) Calculate  $\Delta S$  for  $N = 80$ , which is the spring equinox in the first year.

(b) Calculate  $\Delta S$  for  $N = 1268$ , which is the summer solstice in the fourth year.

(c) What is the maximum value of  $\Delta S$ ?

(d) Find a value for  $N$  where  $\Delta S$  is equal to 0.

(Modeling) Solve each problem. See Example 8.

73. **Average Monthly Temperature** The average monthly temperature (in °F) in Vancouver, Canada, is shown in the table.

Month	°F	Month	°F
Jan	36	July	64
Feb	39	Aug	63
Mar	43	Sept	57
Apr	48	Oct	50
May	55	Nov	43
June	59	Dec	39

Source: Miller, A. and J. Thompson, *Elements of Meteorology*, 4th Edition, Charles E. Merrill Publishing Co., 1983.


(a) Plot the average monthly temperature over a two-year period letting  $x = 1$  correspond to the month of January during the first year. Do the data seem to indicate a translated sine graph?

(b) The highest average monthly temperature is 64°F in July, and the lowest average monthly temperature is 36°F in January. Their average is 50°F. Graph the data together with the line  $y = 50$ . What does this line represent with regard to temperature in Vancouver?

(c) Approximate the amplitude, period, and phase shift of the translated sine wave.

(d) Determine a function of the form  $f(x) = a \sin b(x - d) + c$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants, that models the data.

(e) Graph  $f$  together with the data on the same coordinate axes. How well does  $f$  model the given data?

 (f) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

74. **Average Monthly Temperature** The average monthly temperature (in °F) in Phoenix, Arizona, is shown in the table.

Month	°F	Month	°F
Jan	51	July	90
Feb	55	Aug	90
Mar	63	Sept	84
Apr	67	Oct	71
May	77	Nov	59
June	86	Dec	52


Source: Miller, A. and J. Thompson, *Elements of Meteorology*, 4th Edition, Charles E. Merrill Publishing Co., 1983.

(a) Predict the average yearly temperature and compare it to the actual value of 70°F.

(b) Plot the average monthly temperature over a two-year period by letting  $x = 1$  correspond to January of the first year.

(c) Determine a function of the form  $f(x) = a \cos b(x - d) + c$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants, that models the data.

(d) Graph  $f$  together with the data on the same coordinate axes. How well does  $f$  model the data?

 (e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

## 6.4 Graphs of the Other Circular Functions

Graphs of the Cosecant and Secant Functions ■ Graphs of the Tangent and Cotangent Functions ■ Addition of Ordinates

**Graphs of the Cosecant and Secant Functions** Since cosecant values are reciprocals of the corresponding sine values, the period of the function  $y = \csc x$  is  $2\pi$ , the same as for  $y = \sin x$ . When  $\sin x = 1$ , the value of  $\csc x$  is also 1, and when  $0 < \sin x < 1$ , then  $\csc x > 1$ . Also, if  $-1 < \sin x < 0$ , then  $\csc x < -1$ . (Verify these statements.) As  $|x|$  approaches 0,  $|\sin x|$  approaches 0, and  $|\csc x|$  gets larger and larger. The graph of  $y = \csc x$  approaches the vertical line  $x = 0$  but never touches it, so the line  $x = 0$  is a *vertical asymptote*. In fact, the lines  $x = n\pi$ , where  $n$  is any integer, are all vertical asymptotes.

Using this information and plotting a few points shows that the graph takes the shape of the solid curve shown in Figure 35. To show how the two graphs are related, the graph of  $y = \sin x$  is shown as a dashed curve.

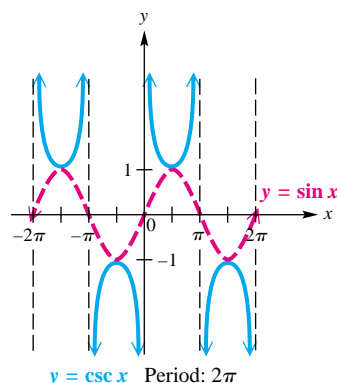


Figure 35

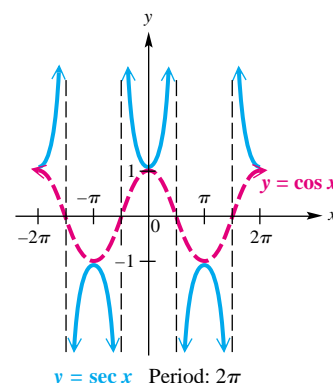


Figure 36

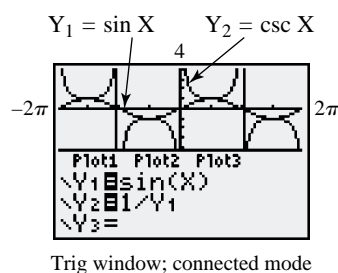


Figure 37

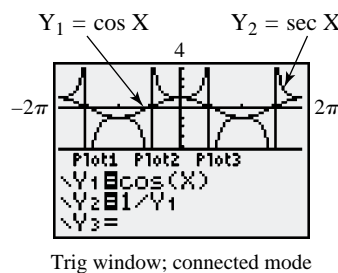


Figure 38

A similar analysis for the secant leads to the solid curve shown in Figure 36. The dashed curve,  $y = \cos x$ , is shown so that the relationship between these two reciprocal functions can be seen.

Typically, calculators do not have keys for the cosecant and secant functions. To graph  $y = \csc x$  with a graphing calculator, use the fact that

$$\csc x = \frac{1}{\sin x}.$$

The graphs of  $Y_1 = \sin X$  and  $Y_2 = \csc X$  are shown in Figure 37. The calculator is in split screen and connected modes. Similarly, the secant function is graphed by using the identity

$$\sec x = \frac{1}{\cos x},$$

as shown in Figure 38.

Using dot mode for graphing will eliminate the vertical lines that appear in Figures 37 and 38. While they suggest asymptotes and are sometimes called *pseudo-asymptotes*, they are not actually parts of the graphs. See Figure 39 on the next page, for example. ■

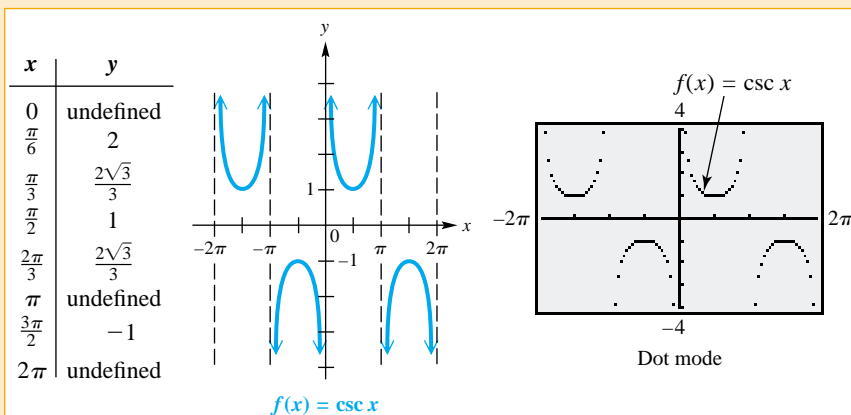
**COSECANT FUNCTION**  $f(x) = \csc x$ Domain:  $\{x \mid x \neq n\pi, \text{ where } n \text{ is an integer}\}$ Range:  $(-\infty, -1] \cup [1, \infty)$ 

Figure 39

- The graph is discontinuous at values of  $x$  of the form  $x = n\pi$  and has vertical asymptotes at these values.
- There are no  $x$ -intercepts.
- Its period is  $2\pi$ .
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all  $x$  in the domain,  $\csc(-x) = -\csc x$ .

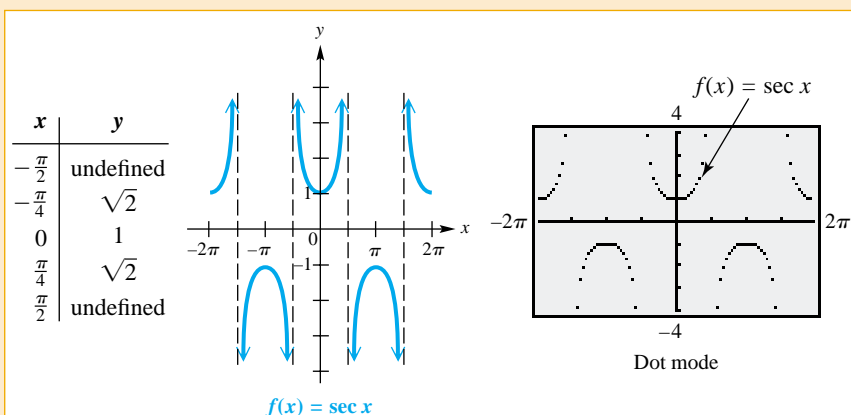
**SECANT FUNCTION**  $f(x) = \sec x$ Domain:  $\{x \mid x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is an integer}\}$ Range:  $(-\infty, -1] \cup [1, \infty)$ 

Figure 40

(continued)

- The graph is discontinuous at values of  $x$  of the form  $x = (2n + 1)\frac{\pi}{2}$  and has vertical asymptotes at these values.
- There are no  $x$ -intercepts.
- Its period is  $2\pi$ .
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the  $y$ -axis, so the function is an even function. For all  $x$  in the domain,  $\sec(-x) = \sec x$ .

In the previous section, we gave guidelines for sketching graphs of sine and cosine functions. We now present similar guidelines for graphing cosecant and secant functions.

### Guidelines for Sketching Graphs of Cosecant and Secant Functions

To graph  $y = a \csc bx$  or  $y = a \sec bx$ , with  $b > 0$ , follow these steps.

*Step 1* Graph the corresponding reciprocal function as a guide, using a dashed curve.

To Graph	Use as a Guide
$y = a \csc bx$	$y = a \sin bx$
$y = a \sec bx$	$y = a \cos bx$

*Step 2* Sketch the vertical asymptotes. They will have equations of the form  $x = k$ , where  $k$  is an  $x$ -intercept of the graph of the guide function.

*Step 3* Sketch the graph of the desired function by drawing the typical U-shaped branches between the adjacent asymptotes. The branches will be above the graph of the guide function when the guide function values are positive and below the graph of the guide function when the guide function values are negative. The graph will resemble those in Figures 39 and 40 in the function boxes on the previous page.

Like graphs of the sine and cosine functions, graphs of the secant and cosecant functions may be translated vertically and horizontally. The period of both basic functions is  $2\pi$ .

**EXAMPLE 1** Graphing  $y = a \sec bx$ 

Graph  $y = 2 \sec \frac{1}{2}x$ .

**Solution**

**Step 1** This function involves the secant, so the corresponding reciprocal function will involve the cosine. The guide function to graph is

$$y = 2 \cos \frac{1}{2}x.$$

Using the guidelines of Section 6.3, we find that this guide function has amplitude 2 and one period of the graph lies along the interval that satisfies the inequality

$$0 \leq \frac{1}{2}x \leq 2\pi, \quad \text{or} \quad [0, 4\pi]. \quad (\text{Section 1.7})$$

Dividing this interval into four equal parts gives the key points

$$(0, 2), \quad (\pi, 0), \quad (2\pi, -2), \quad (3\pi, 0), \quad (4\pi, 2),$$

which are joined with a smooth dashed curve to indicate that this graph is only a guide. An additional period is graphed as seen in Figure 41(a).

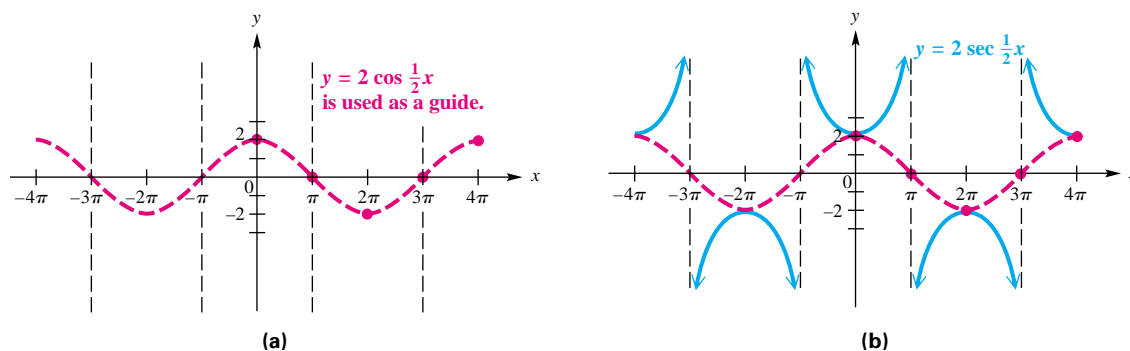
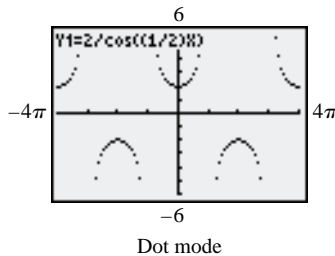


Figure 41



This is a calculator graph of the function in Example 1.

**Step 2** Sketch the vertical asymptotes. These occur at  $x$ -values for which the guide function equals 0, such as

$$x = -3\pi, \quad x = -\pi, \quad x = \pi, \quad x = 3\pi.$$

See Figure 41(a).

**Step 3** Sketch the graph of  $y = 2 \sec \frac{1}{2}x$  by drawing the typical U-shaped branches, approaching the asymptotes. See Figure 41(b).

Now try Exercise 7.

**EXAMPLE 2** Graphing  $y = a \csc(x - d)$ 

$$\text{Graph } y = \frac{3}{2} \csc\left(x - \frac{\pi}{2}\right).$$

**Solution**

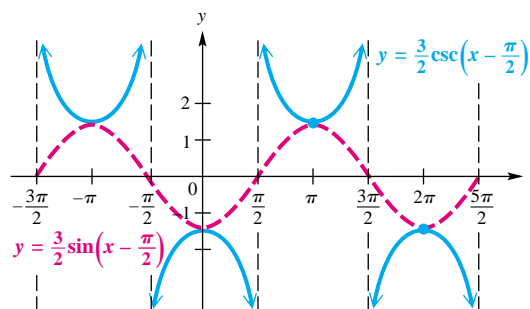
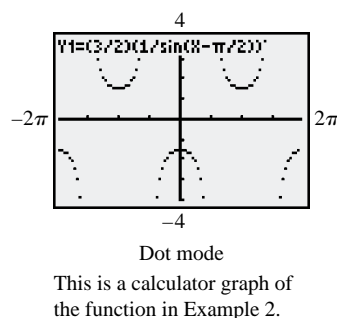
**Step 1** Use the guidelines of Section 6.3 to graph the corresponding reciprocal function

$$y = \frac{3}{2} \sin\left(x - \frac{\pi}{2}\right),$$

shown as a red dashed curve in Figure 42.

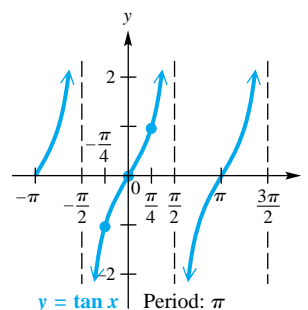
**Step 2** Sketch the vertical asymptotes through the  $x$ -intercepts of the graph of  $y = \frac{3}{2} \sin\left(x - \frac{\pi}{2}\right)$ . These have the form  $x = (2n + 1)\frac{\pi}{2}$ , where  $n$  is an integer. See the black dashed lines in Figure 42.

**Step 3** Sketch the graph of  $y = \frac{3}{2} \csc\left(x - \frac{\pi}{2}\right)$  by drawing the typical U-shaped branches between adjacent asymptotes. See the solid blue graph in Figure 42.

**Figure 42**

**Now try Exercise 9.**

**Graphs of the Tangent and Cotangent Functions** Unlike the four functions whose graphs we studied previously, the tangent function has period  $\pi$ . Because  $\tan x = \frac{\sin x}{\cos x}$ , tangent values are 0 when sine values are 0, and undefined when cosine values are 0. As  $x$ -values go from  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$ , tangent values go from  $-\infty$  to  $\infty$  and increase throughout the interval. Those same values are repeated as  $x$  goes from  $\frac{\pi}{2}$  to  $\frac{3\pi}{2}$ ,  $\frac{3\pi}{2}$  to  $\frac{5\pi}{2}$ , and so on. The graph of  $y = \tan x$  from  $-\pi$  to  $\frac{3\pi}{2}$  is shown in Figure 43.

**Figure 43**

## 580 CHAPTER 6 The Circular Functions and Their Graphs

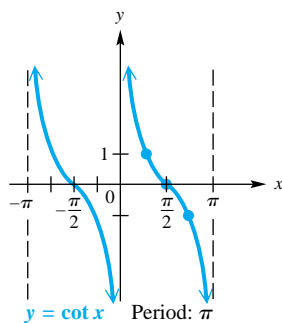


Figure 44

The cotangent function also has period  $\pi$ . Cotangent values are 0 when cosine values are 0, and undefined when sine values are 0. (Verify this also.) As  $x$ -values go from 0 to  $\pi$ , cotangent values go from  $\infty$  to  $-\infty$  and decrease throughout the interval. Those same values are repeated as  $x$  goes from  $\pi$  to  $2\pi$ ,  $2\pi$  to  $3\pi$ , and so on. The graph of  $y = \cot x$  from  $-\pi$  to  $\pi$  is shown in Figure 44.

**TANGENT FUNCTION  $f(x) = \tan x$** 

Domain:  $\{x \mid x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is an integer}\}$  Range:  $(-\infty, \infty)$

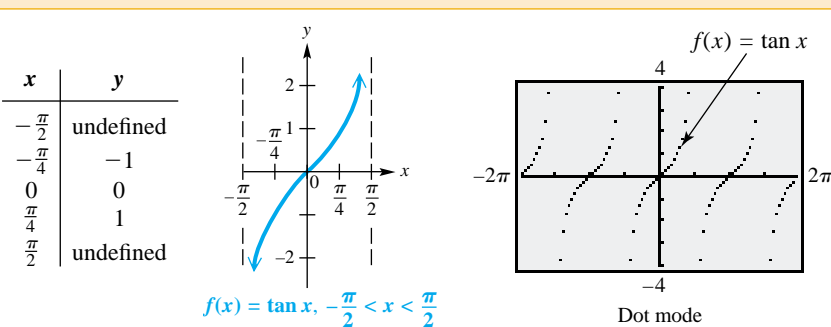


Figure 45

- The graph is discontinuous at values of  $x$  of the form  $x = (2n + 1)\frac{\pi}{2}$  and has vertical asymptotes at these values.
- Its  $x$ -intercepts are of the form  $x = n\pi$ .
- Its period is  $\pi$ .
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all  $x$  in the domain,  $\tan(-x) = -\tan x$ .

**COTANGENT FUNCTION  $f(x) = \cot x$** 

Domain:  $\{x \mid x \neq n\pi, \text{ where } n \text{ is an integer}\}$  Range:  $(-\infty, \infty)$

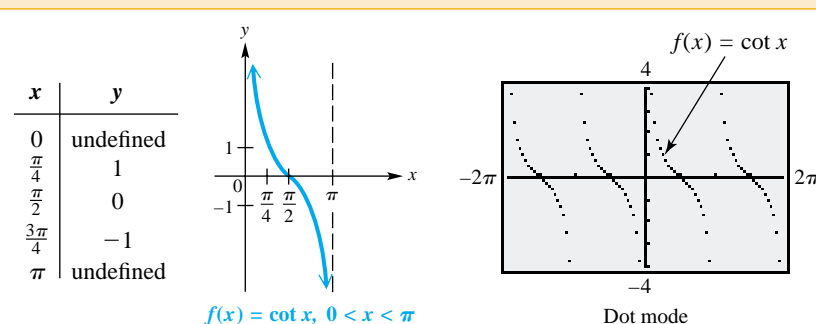


Figure 46

(continued)



- The graph is discontinuous at values of  $x$  of the form  $x = n\pi$  and has vertical asymptotes at these values.
- Its  $x$ -intercepts are of the form  $x = (2n + 1)\frac{\pi}{2}$ .
- Its period is  $\pi$ .
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all  $x$  in the domain,  $\cot(-x) = -\cot x$ .



The tangent function can be graphed directly with a graphing calculator, using the tangent key. To graph the cotangent function, however, we must use one of the identities  $\cot x = \frac{1}{\tan x}$  or  $\cot x = \frac{\cos x}{\sin x}$  since graphing calculators generally do not have cotangent keys. ■

**TEACHING TIP** Emphasize to students that the period of the tangent and cotangent functions is  $\frac{\pi}{b}$ , not  $\frac{2\pi}{b}$ .

### Guidelines for Sketching Graphs of Tangent and Cotangent Functions

To graph  $y = a \tan bx$  or  $y = a \cot bx$ , with  $b > 0$ , follow these steps.

**Step 1** Determine the period,  $\frac{\pi}{b}$ . To locate two adjacent vertical asymptotes, solve the following equations for  $x$ :

$$\text{For } y = a \tan bx: \quad bx = -\frac{\pi}{2} \quad \text{and} \quad bx = \frac{\pi}{2}.$$

$$\text{For } y = a \cot bx: \quad bx = 0 \quad \text{and} \quad bx = \pi.$$

**Step 2** Sketch the two vertical asymptotes found in Step 1.

**Step 3** Divide the interval formed by the vertical asymptotes into four equal parts.

**Step 4** Evaluate the function for the first-quarter point, midpoint, and third-quarter point, using the  $x$ -values found in Step 3.

**Step 5** Join the points with a smooth curve, approaching the vertical asymptotes. Indicate additional asymptotes and periods of the graph as necessary.

### EXAMPLE 3 Graphing $y = \tan bx$

Graph  $y = \tan 2x$ .

#### Solution

**Step 1** The period of this function is  $\frac{\pi}{2}$ . To locate two adjacent vertical asymptotes, solve  $2x = -\frac{\pi}{2}$  and  $2x = \frac{\pi}{2}$  (since this is a tangent function). The two asymptotes have equations  $x = -\frac{\pi}{4}$  and  $x = \frac{\pi}{4}$ .

**Step 2** Sketch the two vertical asymptotes  $x = \pm\frac{\pi}{4}$ , as shown in Figure 47 on the next page.

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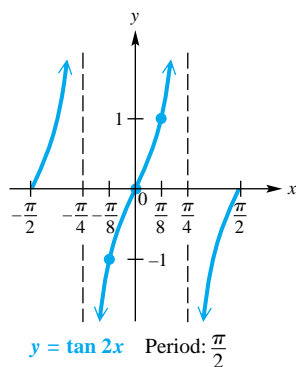


Figure 47

**Step 3** Divide the interval  $(-\frac{\pi}{4}, \frac{\pi}{4})$  into four equal parts. This gives the following key  $x$ -values.

first-quarter value:  $-\frac{\pi}{8}$ , middle value:  $0$ , third-quarter value:  $\frac{\pi}{8}$

**Step 4** Evaluate the function for the  $x$ -values found in Step 3.

$x$	$-\frac{\pi}{8}$	$0$	$\frac{\pi}{8}$
$2x$	$-\frac{\pi}{4}$	$0$	$\frac{\pi}{4}$
$\tan 2x$	$-1$	$0$	$1$

**Step 5** Join these points with a smooth curve, approaching the vertical asymptotes. See Figure 47. Another period has been graphed, one half period to the left and one half period to the right.

Now try Exercise 21.

#### EXAMPLE 4 Graphing $y = a \tan bx$

Graph  $y = -3 \tan \frac{1}{2}x$ .

**Solution** The period is  $\frac{\pi}{\frac{1}{2}} = 2\pi$ . Adjacent asymptotes are at  $x = -\pi$  and  $x = \pi$ . Dividing the interval  $-\pi < x < \pi$  into four equal parts gives key  $x$ -values of  $-\frac{\pi}{2}$ ,  $0$ , and  $\frac{\pi}{2}$ . Evaluating the function at these  $x$ -values gives the key points.

$$\left(-\frac{\pi}{2}, 3\right), \quad (0, 0), \quad \left(\frac{\pi}{2}, -3\right)$$

By plotting these points and joining them with a smooth curve, we obtain the graph shown in Figure 48. Because the coefficient  $-3$  is negative, the graph is reflected across the  $x$ -axis compared to the graph of  $y = 3 \tan \frac{1}{2}x$ .

Now try Exercise 29.

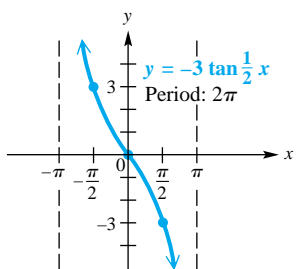


Figure 48

**NOTE** The function defined by  $y = -3 \tan \frac{1}{2}x$  in Example 4, graphed in Figure 48, has a graph that compares to the graph of  $y = \tan x$  as follows.

1. The period is larger because  $b = \frac{1}{2}$ , and  $\frac{1}{2} < 1$ .
2. The graph is “stretched” because  $a = -3$ , and  $|-3| > 1$ .
3. Each branch of the graph goes down from left to right (that is, the function decreases) between each pair of adjacent asymptotes because  $a = -3 < 0$ . When  $a < 0$ , the graph is reflected across the  $x$ -axis compared to the graph of  $y = |a| \tan bx$ .

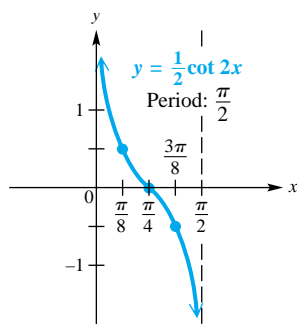


Figure 49

**EXAMPLE 5** Graphing  $y = a \cot bx$ 

Graph  $y = \frac{1}{2} \cot 2x$ .

**Solution** Because this function involves the cotangent, we can locate two adjacent asymptotes by solving the equations  $2x = 0$  and  $2x = \pi$ . The lines  $x = 0$  (the  $y$ -axis) and  $x = \frac{\pi}{2}$  are two such asymptotes. Divide the interval  $(0, \frac{\pi}{2})$  into four equal parts, getting key  $x$ -values of  $\frac{\pi}{8}$ ,  $\frac{\pi}{4}$ , and  $\frac{3\pi}{8}$ . Evaluating the function at these  $x$ -values gives the following key points.

$$\left(\frac{\pi}{8}, \frac{1}{2}\right), \quad \left(\frac{\pi}{4}, 0\right), \quad \left(\frac{3\pi}{8}, -\frac{1}{2}\right)$$

Joining these points with a smooth curve approaching the asymptotes gives the graph shown in Figure 49.

Now try Exercise 31.

Like the other circular functions, the graphs of the tangent and cotangent functions may be translated horizontally and vertically.

**EXAMPLE 6** Graphing a Tangent Function with a Vertical Translation

Graph  $y = 2 + \tan x$ .

**Analytic Solution**

Every value of  $y$  for this function will be 2 units more than the corresponding value of  $y$  in  $y = \tan x$ , causing the graph of  $y = 2 + \tan x$  to be translated 2 units up compared with the graph of  $y = \tan x$ . See Figure 50.

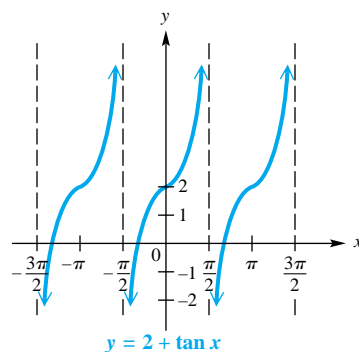


Figure 50

**Graphing Calculator Solution**

To see the vertical translation, observe the coordinates displayed at the bottoms of the screens in Figures 51 and 52. For  $X = \frac{\pi}{4} \approx .78539816$ ,

$$Y_1 = \tan X = 1,$$

while for the same  $X$ -value,

$$Y_2 = 2 + \tan X = 2 + 1 = 3.$$

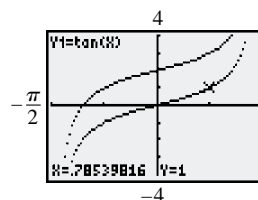


Figure 51

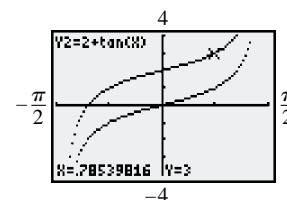


Figure 52

Now try Exercise 37.

**EXAMPLE 7** Graphing a Cotangent Function with Vertical and Horizontal Translations

Graph  $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$ .

**Solution** Here  $b = 1$ , so the period is  $\pi$ . The graph will be translated down 2 units (because  $c = -2$ ), reflected across the  $x$ -axis (because of the negative sign in front of the cotangent), and will have a phase shift (horizontal translation)  $\frac{\pi}{4}$  unit to the right (because of the argument  $(x - \frac{\pi}{4})$ ). To locate adjacent asymptotes, since this function involves the cotangent, we solve the following equations:

$$x - \frac{\pi}{4} = 0, \quad \text{so } x = \frac{\pi}{4} \quad \text{and} \quad x - \frac{\pi}{4} = \pi, \quad \text{so } x = \frac{5\pi}{4}.$$

Dividing the interval  $\frac{\pi}{4} < x < \frac{5\pi}{4}$  into four equal parts and evaluating the function at the three key  $x$ -values within the interval gives these points.

$$\left(\frac{\pi}{2}, -3\right), \quad \left(\frac{3\pi}{4}, -2\right), \quad (\pi, -1)$$

Join these points with a smooth curve. This period of the graph, along with the one in the domain interval  $(-\frac{3\pi}{4}, \frac{\pi}{4})$ , is shown in Figure 53.

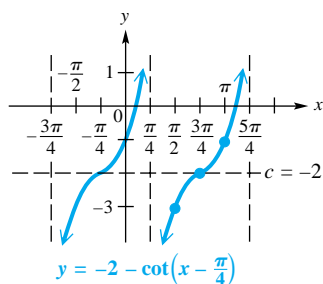


Figure 53

Now try Exercise 45.

**Addition of Ordinates** New functions can be formed by adding or subtracting other functions. A function formed by combining two other functions, such as

$$y = \cos x + \sin x,$$

has historically been graphed using a method known as *addition of ordinates*. (The  $x$ -value of a point is sometimes called its *abscissa*, while its  $y$ -value is called its *ordinate*.) To apply this method to this function, we graph the functions  $y = \cos x$  and  $y = \sin x$ . Then, for selected values of  $x$ , we add  $\cos x$  and  $\sin x$ , and plot the points  $(x, \cos x + \sin x)$ . Joining the resulting points with a sinusoidal curve gives the graph of the desired function. While this method illustrates some valuable concepts involving the arithmetic of functions, it is time-consuming.

With graphing calculators, this technique is easily illustrated. Let  $Y_1 = \cos X$ ,  $Y_2 = \sin X$ , and  $Y_3 = Y_1 + Y_2$ . Figure 54 shows the result when  $Y_1$  and  $Y_2$  are graphed in thin graph style, and  $Y_3 = \cos X + \sin X$  is graphed in thick graph style. Notice that for  $X = \frac{\pi}{6} \approx .52359878$ ,  $Y_1 + Y_2 = Y_3$ .

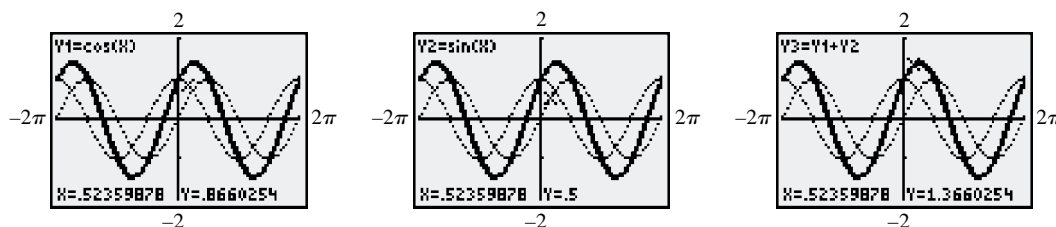
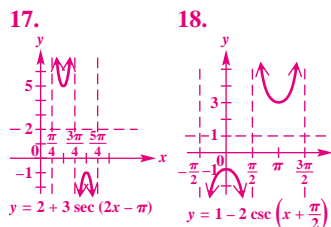
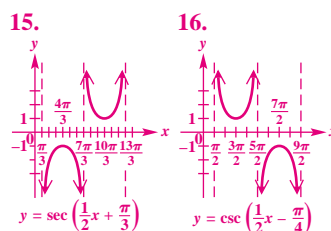
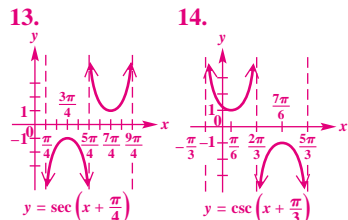
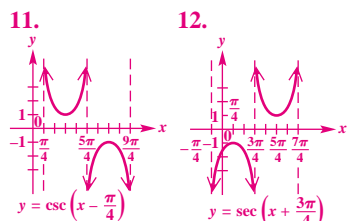
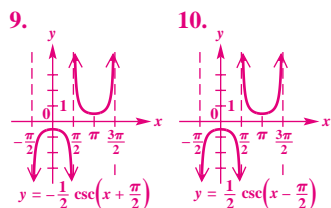
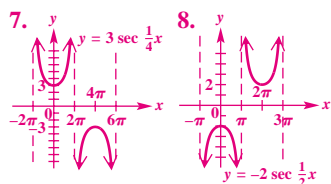


Figure 54

Now try Exercise 61.

## 6.4 Exercises

1. B 2. C 3. E 4. A 5. D  
6. F**Concept Check** In Exercises 1–6, match each function with its graph from choices A–F.

1.  $y = -\csc x$

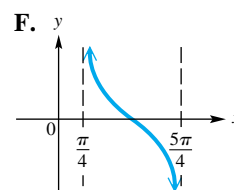
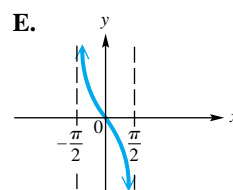
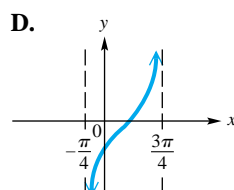
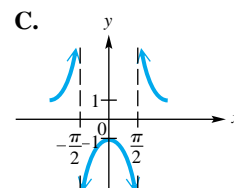
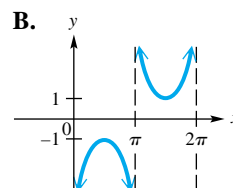
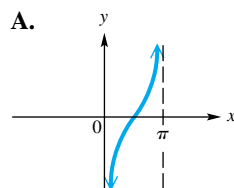
2.  $y = -\sec x$

3.  $y = -\tan x$

4.  $y = -\cot x$

5.  $y = \tan\left(x - \frac{\pi}{4}\right)$

6.  $y = \cot\left(x - \frac{\pi}{4}\right)$



Graph each function over a one-period interval. See Examples 1 and 2.

7.  $y = 3 \sec \frac{1}{4}x$

8.  $y = -2 \sec \frac{1}{2}x$

9.  $y = -\frac{1}{2} \csc\left(x + \frac{\pi}{2}\right)$

10.  $y = \frac{1}{2} \csc\left(x - \frac{\pi}{2}\right)$

11.  $y = \csc\left(x - \frac{\pi}{4}\right)$

12.  $y = \sec\left(x + \frac{3\pi}{4}\right)$

13.  $y = \sec\left(x + \frac{\pi}{4}\right)$

14.  $y = \csc\left(x + \frac{\pi}{3}\right)$

15.  $y = \sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

16.  $y = \csc\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

17.  $y = 2 + 3 \sec(2x - \pi)$

18.  $y = 1 - 2 \csc\left(x + \frac{\pi}{2}\right)$

19.  $y = 1 - \frac{1}{2} \csc\left(x - \frac{3\pi}{4}\right)$

20.  $y = 2 + \frac{1}{4} \sec\left(\frac{1}{2}x - \pi\right)$

Graph each function over a one-period interval. See Examples 3–5.

21.  $y = \tan 4x$

22.  $y = \tan \frac{1}{2}x$

23.  $y = 2 \tan x$

24.  $y = 2 \cot x$

25.  $y = 2 \tan \frac{1}{4}x$

26.  $y = \frac{1}{2} \cot x$

27.  $y = \cot 3x$

28.  $y = -\cot \frac{1}{2}x$

29.  $y = -2 \tan \frac{1}{4}x$

30.  $y = 3 \tan \frac{1}{2}x$

31.  $y = \frac{1}{2} \cot 4x$

32.  $y = -\frac{1}{2} \cot 2x$

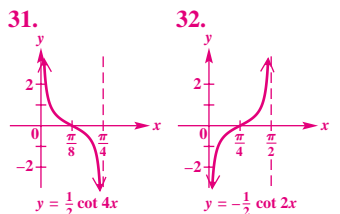
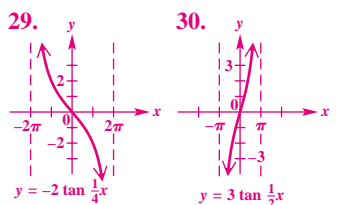
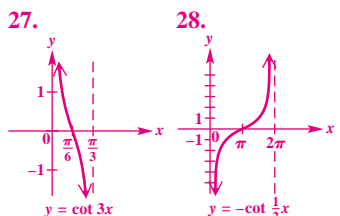
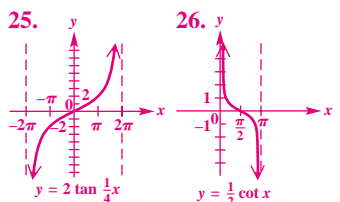
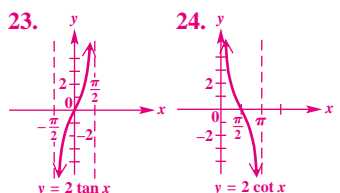
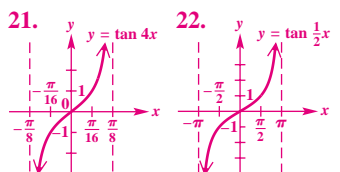
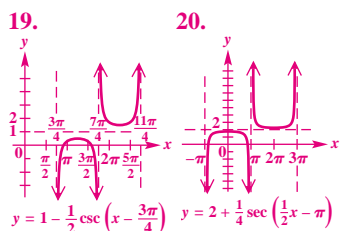
Graph each function over a two-period interval. See Examples 6 and 7.

33.  $y = \tan(2x - \pi)$

34.  $y = \tan\left(\frac{x}{2} + \pi\right)$

35.  $y = \cot\left(3x + \frac{\pi}{4}\right)$

## 586 CHAPTER 6 The Circular Functions and Their Graphs



Answer graphs for odd-numbered Exercises 33–45 are included on page A-39 of the answer section at the back of the text.

36.  $y = \cot\left(2x - \frac{3\pi}{2}\right)$     37.  $y = 1 + \tan x$     38.  $y = 1 - \tan x$
39.  $y = 1 - \cot x$     40.  $y = -2 - \cot x$
41.  $y = -1 + 2 \tan x$     42.  $y = 3 + \frac{1}{2} \tan x$
43.  $y = -1 + \frac{1}{2} \cot(2x - 3\pi)$     44.  $y = -2 + 3 \tan(4x + \pi)$
45.  $y = 1 - 2 \cot 2\left(x + \frac{\pi}{2}\right)$     46.  $y = \frac{2}{3} \tan\left(\frac{3}{4}x - \pi\right) - 2$

**Concept Check** In Exercises 47–52, tell whether each statement is true or false. If false, tell why.

47. The smallest positive number  $k$  for which  $x = k$  is an asymptote for the tangent function is  $\frac{\pi}{2}$ .
48. The smallest positive number  $k$  for which  $x = k$  is an asymptote for the cotangent function is  $\frac{\pi}{2}$ .
49. The tangent and secant functions are undefined for the same values.
50. The secant and cosecant functions are undefined for the same values.
51. The graph of  $y = \tan x$  in Figure 45 suggests that  $\tan(-x) = \tan x$  for all  $x$  in the domain of  $\tan x$ .
52. The graph of  $y = \sec x$  in Figure 40 suggests that  $\sec(-x) = \sec x$  for all  $x$  in the domain of  $\sec x$ .
53. **Concept Check** If  $c$  is any number, then how many solutions does the equation  $c = \tan x$  have in the interval  $(-2\pi, 2\pi]$ ?
54. **Concept Check** If  $c$  is any number such that  $-1 < c < 1$ , then how many solutions does the equation  $c = \sec x$  have over the entire domain of the secant function?
55. Consider the function defined by  $f(x) = -4 \tan(2x + \pi)$ . What is the domain of  $f$ ? What is its range?
56. Consider the function defined by  $g(x) = -2 \csc(4x + \pi)$ . What is the domain of  $g$ ? What is its range?

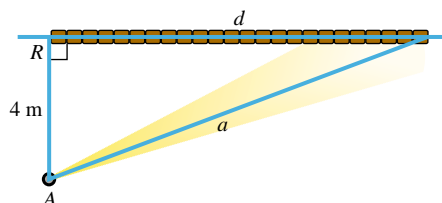
**(Modeling)** Solve each problem.

57. **Distance of a Rotating Beacon** A rotating beacon is located at point A next to a long wall. (See the figure.) The beacon is 4 m from the wall. The distance  $d$  is given by

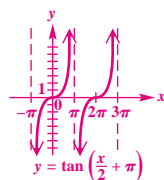
$$d = 4 \tan 2\pi t,$$

where  $t$  is time measured in seconds since the beacon started rotating. (When  $t = 0$ , the beacon is aimed at point R. When the beacon is aimed to the right of R, the value of  $d$  is positive;  $d$  is negative if the beacon is aimed to the left of R.) Find  $d$  for each time.

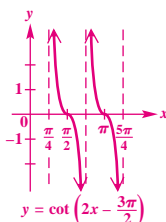
- (a)  $t = 0$   
 (b)  $t = .4$   
 (c)  $t = .8$   
 (d)  $t = 1.2$   
 (e) Why is .25 a meaningless value for  $t$ ?



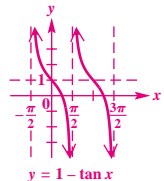
34.



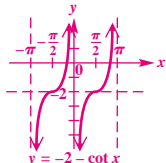
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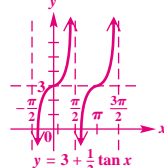
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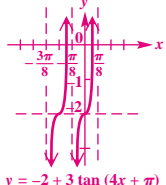
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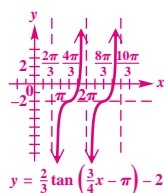
42.



44.



46.



47. true 48. false; The smallest such  $k$  is  $\pi$ . 49. true 50. false; Secant values are undefined when  $x = \frac{\pi}{2} + n\pi$ , while cosecant values are undefined when  $x = n\pi$ . 51. false;  $\tan(-x) = -\tan x$  for all  $x$  in the domain. 52. true 53. four 54. none

58. **Distance of a Rotating Beacon** In the figure for Exercise 57, the distance  $a$  is given by

$$a = 4|\sec 2\pi t|.$$

Find  $a$  for each time.

(a)  $t = 0$ (b)  $t = .86$ (c)  $t = 1.24$ 

59. Simultaneously graph  $y = \tan x$  and  $y = x$  in the window  $[-1, 1]$  by  $[-1, 1]$  with a graphing calculator. Write a sentence or two describing the relationship of  $\tan x$  and  $x$  for small  $x$ -values.



60. Between each pair of successive asymptotes, a portion of the graph of  $y = \sec x$  or  $y = \csc x$  resembles a parabola. Can each of these portions actually be a parabola? Explain.



Use a graphing calculator to graph  $Y_1$ ,  $Y_2$ , and  $Y_1 + Y_2$  on the same screen. Evaluate each of the three functions at  $X = \frac{\pi}{6}$ , and verify that  $Y_1\left(\frac{\pi}{6}\right) + Y_2\left(\frac{\pi}{6}\right) = (Y_1 + Y_2)\left(\frac{\pi}{6}\right)$ . See the discussion on addition of ordinates.

61.  $Y_1 = \sin X$ ,  $Y_2 = \sin 2X$ 62.  $Y_1 = \cos X$ ,  $Y_2 = \sec X$ 

## Relating Concepts

For individual or collaborative investigation

(Exercises 63–68)

Consider the function defined by  $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$  from Example 7. **Work these exercises in order.**

63. What is the smallest positive number for which  $y = \cot x$  is undefined?

64. Let  $k$  represent the number you found in Exercise 63. Set  $x - \frac{\pi}{4}$  equal to  $k$ , and solve to find the smallest positive number for which  $\cot\left(x - \frac{\pi}{4}\right)$  is undefined.

65. Based on your answer in Exercise 64 and the fact that the cotangent function has period  $\pi$ , give the general form of the equations of the asymptotes of the graph of  $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$ . Let  $n$  represent any integer.



66. Use the capabilities of your calculator to find the smallest positive  $x$ -intercept of the graph of this function.

67. Use the fact that the period of this function is  $\pi$  to find the next positive  $x$ -intercept.

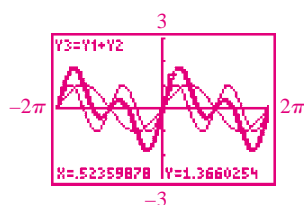
68. Give the solution set of the equation  $-2 - \cot\left(x - \frac{\pi}{4}\right) = 0$  over all real numbers. Let  $n$  represent any integer.

55. domain:  $\left\{x \mid x \neq (2n + 1)\frac{\pi}{4}, \text{ where } n \text{ is an integer}\right\}$ ; range:  $(-\infty, \infty)$  56. domain:  $\left\{x \mid x \neq \frac{n\pi}{4}, \text{ where } n \text{ is an integer}\right\}$ ;

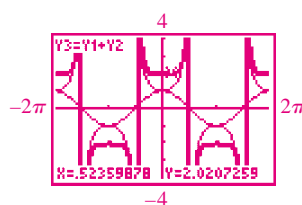
range:  $(-\infty, -2] \cup [2, \infty)$  57. (a) 0 m (b) -2.9 m (c) -12.3 m (d) 12.3 m (e) It leads to  $\tan \frac{\pi}{2}$ , which is undefined.

58. (a) 4 m (b) 6.3 m (c) 63.7 m In Exercises 61 and 62, we show the display for  $Y_1 + Y_2$  at  $X = \frac{\pi}{6}$ .

61.



62.

63.  $\pi$  64.  $\frac{5\pi}{4}$  65.  $y = \frac{5\pi}{4} + n\pi$ 

66. approximately .3217505544

67. approximately 3.463343208

68.  $\{x \mid x = .3217505544 + n\pi\}$

## Summary Exercises on Graphing Circular Functions

Answer graphs for the Summary Exercises are given on page A-39 of the answer section at the back of the text.

These summary exercises provide practice with the various graphing techniques presented in this chapter. Graph each function over a one-period interval.

1.  $y = 2 \sin \pi x$

2.  $y = 4 \cos 1.5x$

3.  $y = -2 + .5 \cos \frac{\pi}{4}(x - 1)$

4.  $y = 3 \sec \frac{\pi x}{2}$

5.  $y = -4 \csc .5x$

6.  $y = 3 \tan\left(\frac{\pi x}{2} + \pi\right)$

Graph each function over a two-period interval.

7.  $y = -5 \sin \frac{x}{3}$

8.  $y = 10 \cos\left(\frac{x}{4} + \frac{\pi}{2}\right)$

9.  $y = 3 - 4 \sin(2.5x + \pi)$

10.  $y = 2 - \sec[\pi(x - 3)]$

## 6.5 Harmonic Motion

Simple Harmonic Motion ■ Damped Oscillatory Motion

**Simple Harmonic Motion** In part A of Figure 55, a spring with a weight attached to its free end is in equilibrium (or rest) position. If the weight is pulled down  $a$  units and released (part B of the figure), the spring's elasticity causes the weight to rise  $a$  units ( $a > 0$ ) above the equilibrium position, as seen in part C, and then oscillate about the equilibrium position. If friction is neglected, this oscillatory motion is described mathematically by a sinusoid. Other applications of this type of motion include sound, electric current, and electromagnetic waves.

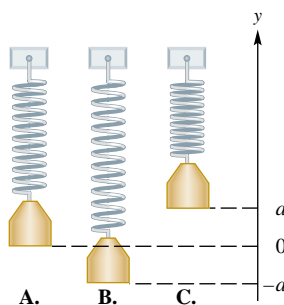


Figure 55

To develop a general equation for such motion, consider Figure 56. Suppose the point  $P(x, y)$  moves around the circle counterclockwise at a uniform angular speed  $\omega$ . Assume that at time  $t = 0$ ,  $P$  is at  $(a, 0)$ . The angle swept out by ray  $OP$  at time  $t$  is given by  $\theta = \omega t$ . The coordinates of point  $P$  at time  $t$  are

$$x = a \cos \theta = a \cos \omega t \quad \text{and} \quad y = a \sin \theta = a \sin \omega t.$$

As  $P$  moves around the circle from the point  $(a, 0)$ , the point  $Q(0, y)$  oscillates back and forth along the  $y$ -axis between the points  $(0, a)$  and  $(0, -a)$ . Similarly, the point  $R(x, 0)$  oscillates back and forth between  $(a, 0)$  and  $(-a, 0)$ . This oscillatory motion is called **simple harmonic motion**.

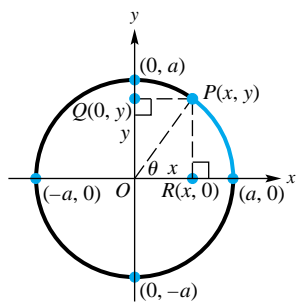


Figure 56



The amplitude of the motion is  $|a|$ , and the period is  $\frac{2\pi}{\omega}$ . The moving points  $P$  and  $Q$  or  $P$  and  $R$  complete one oscillation or cycle per period. The number of cycles per unit of time, called the **frequency**, is the reciprocal of the period,  $\frac{\omega}{2\pi}$ , where  $\omega > 0$ .

**TEACHING TIP** Mention that simple harmonic motion is an important topic for students majoring in engineering or physics.

### Simple Harmonic Motion

The position of a point oscillating about an equilibrium position at time  $t$  is modeled by either

$$s(t) = a \cos \omega t \quad \text{or} \quad s(t) = a \sin \omega t,$$

where  $a$  and  $\omega$  are constants, with  $\omega > 0$ . The amplitude of the motion is  $|a|$ , the period is  $\frac{2\pi}{\omega}$ , and the frequency is  $\frac{\omega}{2\pi}$ .

#### EXAMPLE 1 Modeling the Motion of a Spring

Suppose that an object is attached to a coiled spring such as the one in Figure 55. It is pulled down a distance of 5 in. from its equilibrium position, and then released. The time for one complete oscillation is 4 sec.

- Give an equation that models the position of the object at time  $t$ .
- Determine the position at  $t = 1.5$  sec.
- Find the frequency.

#### Solution

- When the object is released at  $t = 0$ , the distance of the object from the equilibrium position is 5 in. below equilibrium. If  $s(t)$  is to model the motion, then  $s(0)$  must equal  $-5$ . We use

$$s(t) = a \cos \omega t,$$

with  $a = -5$ . We choose the cosine function because  $\cos \omega(0) = \cos 0 = 1$ , and  $-5 \cdot 1 = -5$ . (Had we chosen the sine function, a phase shift would have been required.) The period is 4, so

$$\frac{2\pi}{\omega} = 4, \quad \text{or} \quad \omega = \frac{\pi}{2}. \quad \text{Solve for } \omega. \text{ (Section 1.1)}$$

Thus, the motion is modeled by

$$s(t) = -5 \cos \frac{\pi}{2} t.$$

- After 1.5 sec, the position is

$$s(1.5) = -5 \cos \left[ \frac{\pi}{2} (1.5) \right] \approx 3.54 \text{ in.}$$

Since  $3.54 > 0$ , the object is above the equilibrium position.

- The frequency is the reciprocal of the period, or  $\frac{1}{4}$ .

Now try Exercise 9.

**EXAMPLE 2** Analyzing Harmonic Motion

Suppose that an object oscillates according to the model

$$s(t) = 8 \sin 3t,$$

where  $t$  is in seconds and  $s(t)$  is in feet. Analyze the motion.

**Solution** The motion is harmonic because the model is of the form  $s(t) = a \sin \omega t$ . Because  $a = 8$ , the object oscillates 8 ft in either direction from its starting point. The period  $\frac{2\pi}{3} \approx 2.1$  is the time, in seconds, it takes for one complete oscillation. The frequency is the reciprocal of the period, so the object completes  $\frac{3}{2\pi} \approx .48$  oscillation per sec.

Now try Exercise 15.

**Damped Oscillatory Motion** In the example of the stretched spring, we disregard the effect of friction. Friction causes the amplitude of the motion to diminish gradually until the weight comes to rest. In this situation, we say that the motion has been *damped* by the force of friction. Most oscillatory motions are damped, and the decrease in amplitude follows the pattern of exponential decay. A typical example of **damped oscillatory motion** is provided by the function defined by

$$s(t) = e^{-t} \sin t.$$

Figure 57 shows how the graph of  $y_3 = e^{-x} \sin x$  is bounded above by the graph of  $y_1 = e^{-x}$  and below by the graph of  $y_2 = -e^{-x}$ . The damped motion curve dips below the  $x$ -axis at  $x = \pi$  but stays above the graph of  $y_2$ . Figure 58 shows a traditional graph of  $s(t) = e^{-t} \sin t$ , along with the graph of  $y = \sin t$ .

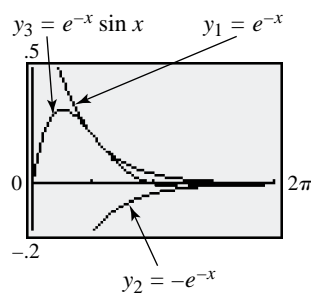


Figure 57

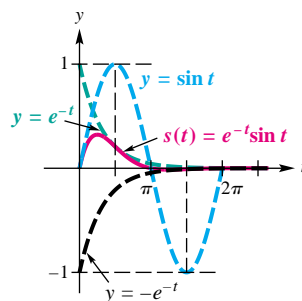


Figure 58

Shock absorbers are put on an automobile in order to damp oscillatory motion. Instead of oscillating up and down for a long while after hitting a bump or pothole, the oscillations of the car are quickly damped out for a smoother ride.

Now try Exercise 21.

## 6.5 Exercises

1. (a)  $s(t) = 2 \cos 4\pi t$   
 (b)  $s(1) = 2$ ; The weight is neither moving upward nor downward. At  $t = 1$ , the motion of the weight is changing from up to down.

2. (a)  $s(t) = 5 \cos \frac{4\pi}{3} t$

- (b)  $s(1) = -2.5$ ; upward

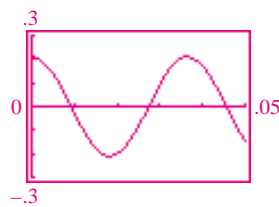
3. (a)  $s(t) = -3 \cos 2.5\pi t$

- (b)  $s(1) = 0$ ; upward

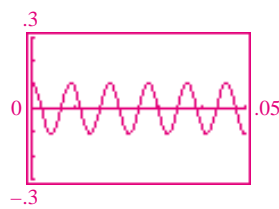
4. (a)  $s(t) = -4 \cos \frac{5\pi}{3} t$

- (b)  $s(1) = -2$ ; downward

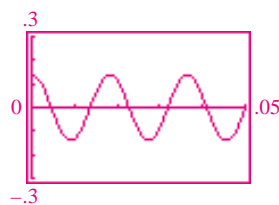
5.  $s(t) = .21 \cos 55\pi t$



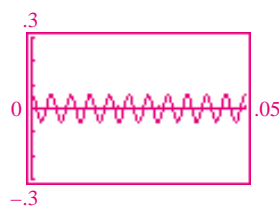
6.  $s(t) = .11 \cos 220\pi t$



7.  $s(t) = .14 \cos 110\pi t$



8.  $s(t) = .06 \cos 440\pi t$



9. (a)  $s(t) = -4 \cos \frac{2\pi}{3} t$

- (b) 3.46 in. (c)  $\frac{1}{3}$

**(Modeling) Springs** Suppose that a weight on a spring has initial position  $s(0)$  and period  $P$ .

- (a) Find a function  $s$  given by  $s(t) = a \cos \omega t$  that models the displacement of the weight.

- (b) Evaluate  $s(1)$ . Is the weight moving upward, downward, or neither when  $t = 1$ ? Support your results graphically or numerically.

1.  $s(0) = 2$  in.;  $P = .5$  sec

2.  $s(0) = 5$  in.;  $P = 1.5$  sec

3.  $s(0) = -3$  in.;  $P = .8$  sec

4.  $s(0) = -4$  in.;  $P = 1.2$  sec



**(Modeling) Music** A note on the piano has given frequency  $F$ . Suppose the maximum displacement at the center of the piano wire is given by  $s(t)$ . Find constants  $a$  and  $\omega$  so that the equation  $s(t) = a \cos \omega t$  models this displacement. Graph  $s$  in the viewing window  $[0, .05]$  by  $[-.3, .3]$ .

5.  $F = 27.5$ ;  $s(0) = .21$

6.  $F = 110$ ;  $s(0) = .11$

7.  $F = 55$ ;  $s(0) = .14$

8.  $F = 220$ ;  $s(0) = .06$

**(Modeling)** Solve each problem. See Examples 1 and 2.

9. **Spring** An object is attached to a coiled spring, as in Figure 55. It is pulled down a distance of 4 units from its equilibrium position, and then released. The time for one complete oscillation is 3 sec.

- (a) Give an equation that models the position of the object at time  $t$ .

- (b) Determine the position at  $t = 1.25$  sec.

- (c) Find the frequency.

10. **Spring** Repeat Exercise 9, but assume that the object is pulled down 6 units and the time for one complete oscillation is 4 sec.

11. **Particle Movement** Write the equation and then determine the amplitude, period, and frequency of the simple harmonic motion of a particle moving uniformly around a circle of radius 2 units, with angular speed

- (a) 2 radians per sec

- (b) 4 radians per sec.

12. **Pendulum** What are the period  $P$  and frequency  $T$  of oscillation of a pendulum of length  $\frac{1}{2}$  ft? (Hint:  $P = 2\pi\sqrt{\frac{L}{32}}$ , where  $L$  is the length of the pendulum in feet and  $P$  is in seconds.)

13. **Pendulum** In Exercise 12, how long should the pendulum be to have period 1 sec?

14. **Spring** The formula for the up and down motion of a weight on a spring is given by

$$s(t) = a \sin \sqrt{\frac{k}{m}} t.$$

If the spring constant  $k$  is 4, what mass  $m$  must be used to produce a period of 1 sec?

15. **Spring** The height attained by a weight attached to a spring set in motion is

$$s(t) = -4 \cos 8\pi t$$

inches after  $t$  seconds.

- (a) Find the maximum height that the weight rises above the equilibrium position of  $y = 0$ .

- (b) When does the weight first reach its maximum height, if  $t \geq 0$ ?

- (c) What are the frequency and period?

## 592 CHAPTER 6 The Circular Functions and Their Graphs

10. (a)  $s(t) = -6 \cos \frac{\pi}{2}t$

(b) 2.30 in. (c)  $\frac{1}{4}$

11. (a)  $s(t) = 2 \sin 2t$ ;  
amplitude: 2; period:  $\pi$ ;

frequency:  $\frac{1}{\pi}$

(b)  $s(t) = 2 \sin 4t$ ; amplitude: 2;

period:  $\frac{\pi}{2}$ ; frequency:  $\frac{2}{\pi}$

12. period:  $\frac{\pi}{4}$ ; frequency:  $\frac{4}{\pi}$

13.  $\frac{8}{\pi^2}$  14.  $\frac{1}{\pi^2}$

15. (a) 4 in. (b) after  $\frac{1}{8}$  sec

(c) 4 cycles per sec;  $\frac{1}{4}$  sec

16. (a) amplitude:  $\frac{1}{2}$ ;

period:  $\sqrt{2}\pi$ ; frequency:  $\frac{\sqrt{2}}{2\pi}$

(b)  $s(t) = \frac{1}{2} \sin \sqrt{2}t$

17. (a) 5 in. (b) 2 cycles per sec;  
 $\frac{1}{2}$  sec (c) after  $\frac{1}{4}$  sec

(d) approximately 4; After  
1.3 sec, the weight is about 4 in.  
above the equilibrium position.

18. (a) 4 in. (b)  $\frac{5}{\pi}$  cycles

per sec;  $\frac{\pi}{5}$  sec

(c) after  $\frac{\pi}{10}$  sec

(d) approximately 2; After  
1.466 sec, the weight is about 2 in.  
above the equilibrium position.

19. (a)  $s(t) = -3 \cos 12t$

(b)  $\frac{\pi}{6}$  sec

20. (a)  $s(t) = -2 \cos 6\pi t$

(b) 3 cycles per sec 21. 0;  $\pi$ ;  
They are the same. 22. for  $y_1$

and  $y_2$ :  $1.5707963 \approx \frac{\pi}{2}$ ; for

$y_1$  and  $y_3$ : none in  $[0, \pi]$ ;

Because  $\sin \frac{\pi}{2} = 1$ ,

$e^{-\pi/2} \sin \frac{\pi}{2} = e^{-\pi/2}$ .

16. **Spring** (See Exercise 14.) A spring with spring constant  $k = 2$  and a 1-unit mass  $m$  attached to it is stretched and then allowed to come to rest.

(a) If the spring is stretched  $\frac{1}{2}$  ft and released, what are the amplitude, period, and frequency of the resulting oscillatory motion?

(b) What is the equation of the motion?

17. **Spring** The position of a weight attached to a spring is

$$s(t) = -5 \cos 4\pi t$$

inches after  $t$  seconds.

(a) What is the maximum height that the weight rises above the equilibrium position?

(b) What are the frequency and period?

(c) When does the weight first reach its maximum height?

(d) Calculate and interpret  $s(1.3)$ .

18. **Spring** The position of a weight attached to a spring is

$$s(t) = -4 \cos 10t$$

inches after  $t$  seconds.

(a) What is the maximum height that the weight rises above the equilibrium position?

(b) What are the frequency and period?

(c) When does the weight first reach its maximum height?

(d) Calculate and interpret  $s(1.466)$ .

19. **Spring** A weight attached to a spring is pulled down 3 in. below the equilibrium position.


(a) Assuming that the frequency is  $\frac{6}{\pi}$  cycles per sec, determine a model that gives the position of the weight at time  $t$  seconds.

(b) What is the period?

20. **Spring** A weight attached to a spring is pulled down 2 in. below the equilibrium position.

(a) Assuming that the period is  $\frac{1}{3}$  sec, determine a model that gives the position of the weight at time  $t$  seconds.

(b) What is the frequency?

 Use a graphing calculator to graph  $y_1 = e^{-t} \sin t$ ,  $y_2 = e^{-t}$ , and  $y_3 = -e^{-t}$  in the viewing window  $[0, \pi]$  by  $[-.5, .5]$ .

21. Find the  $t$ -intercepts of the graph of  $y_1$ . Explain the relationship of these intercepts with the  $x$ -intercepts of the graph of  $y = \sin x$ .

22. Find any points of intersection of  $y_1$  and  $y_2$  or  $y_1$  and  $y_3$ . How are these points related to the graph of  $y = \sin x$ ?

## Chapter 6 Summary

### KEY TERMS

<b>6.1</b> radian sector of a circle	<b>6.3</b> linear speed $v$ periodic function period sine wave (sinusoid) amplitude	phase shift argument <b>6.4</b> addition of ordinates <b>6.5</b> simple harmonic motion	frequency damped oscillatory motion
<b>6.2</b> unit circle circular functions angular speed $\omega$			

### QUICK REVIEW

#### CONCEPTS

#### 6.1 Radian Measure

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of **1 radian**.

**Degree/Radian Relationship**  $180^\circ = \pi$  radians

#### Converting Between Degrees and Radians

1. Multiply a degree measure by  $\frac{\pi}{180}$  radian and simplify to convert to radians.
2. Multiply a radian measure by  $\frac{180^\circ}{\pi}$  and simplify to convert to degrees.

#### Arc Length

The length  $s$  of the arc intercepted on a circle of radius  $r$  by a central angle of measure  $\theta$  radians is given by the product of the radius and the radian measure of the angle, or

$$s = r\theta, \quad \theta \text{ in radians.}$$

#### Area of a Sector

The area of a sector of a circle of radius  $r$  and central angle  $\theta$  is given by

$$A = \frac{1}{2}r^2\theta, \quad \theta \text{ in radians.}$$

#### EXAMPLES

Convert  $135^\circ$  to radians.

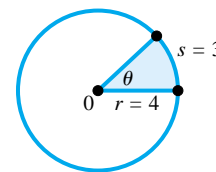
$$135^\circ = 135 \left( \frac{\pi}{180} \text{ radian} \right) = \frac{3\pi}{4} \text{ radians}$$

Convert  $-\frac{5\pi}{3}$  radians to degrees.

$$-\frac{5\pi}{3} \text{ radians} = -\frac{5\pi}{3} \left( \frac{180^\circ}{\pi} \right) = -300^\circ$$

In the figure,  $s = r\theta$  so

$$\theta = \frac{s}{r} = \frac{3}{4} \text{ radian.}$$



The area of the sector in the figure is

$$A = \frac{1}{2}(4)^2 \left( \frac{3}{4} \right) = 6 \text{ square units.}$$

## CONCEPTS

## EXAMPLES

### 6.2 The Unit Circle and Circular Functions

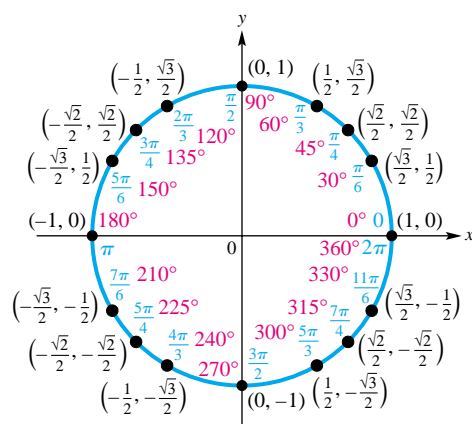
#### Circular Functions

Start at the point  $(1, 0)$  on the unit circle  $x^2 + y^2 = 1$  and lay off an arc of length  $|s|$  along the circle, going counterclockwise if  $s$  is positive, and clockwise if  $s$  is negative. Let the endpoint of the arc be at the point  $(x, y)$ . The six circular functions of  $s$  are defined as follows. (Assume that no denominators are 0.)

$$\sin s = y \quad \cos s = x \quad \tan s = \frac{y}{x}$$

$$\csc s = \frac{1}{y} \quad \sec s = \frac{1}{x} \quad \cot s = \frac{x}{y}$$

#### The Unit Circle



Unit circle  $x^2 + y^2 = 1$

#### Formulas for Angular and Linear Speed

Angular Speed	Linear Speed
$\omega = \frac{\theta}{t}$	$v = \frac{s}{t}$
( $\omega$ in radians per unit time, $\theta$ in radians)	$v = \frac{r\theta}{t}$
	$v = r\omega$

Use the unit circle to find each value.

$$\sin \frac{5\pi}{6} = \frac{1}{2}$$

$$\cos \frac{3\pi}{2} = 0$$

$$\tan \frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\csc \frac{7\pi}{4} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\sec \frac{7\pi}{6} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot \frac{\pi}{3} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

A belt runs a pulley of radius 8 in. at 60 revolutions per min. Find the angular speed  $\omega$  in radians per minute, and the linear speed  $v$  of the belt in inches per minute.

$$\omega = 60(2\pi) = 120\pi \text{ radians per min}$$

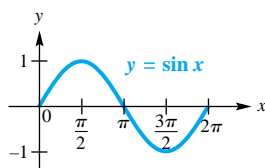
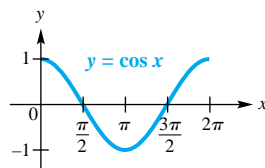
$$v = r\omega = 8(120\pi) = 960\pi \text{ in. per min}$$

## CONCEPTS

## EXAMPLES

## 6.3 Graphs of the Sine and Cosine Functions

## Sine and Cosine Functions

**Domain:**  $(-\infty, \infty)$ **Range:**  $[-1, 1]$ **Amplitude:** 1**Period:**  $2\pi$ **Domain:**  $(-\infty, \infty)$ **Range:**  $[-1, 1]$ **Amplitude:** 1**Period:**  $2\pi$ 

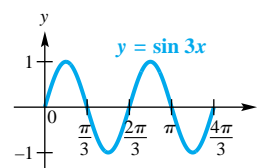
The graph of

$$y = c + a \sin b(x - d) \quad \text{or} \quad y = c + a \cos b(x - d),$$

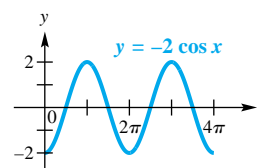
 $b > 0$ , has

1. amplitude  $|a|$ ,
2. period  $\frac{2\pi}{b}$ ,
3. vertical translation  $c$  units up if  $c > 0$  or  $|c|$  units down if  $c < 0$ , and
4. phase shift  $d$  units to the right if  $d > 0$  or  $|d|$  units to the left if  $d < 0$ .

See pages 562 and 565 for a summary of graphing techniques.

Graph  $y = \sin 3x$ .period:  $\frac{2\pi}{3}$ 

amplitude: 1

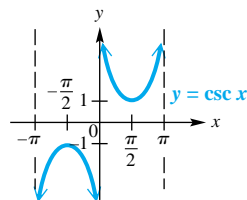
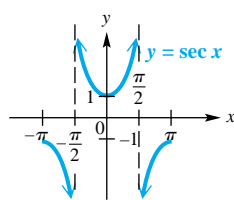
domain:  $(-\infty, \infty)$ range:  $[-1, 1]$ Graph  $y = -2 \cos x$ .period:  $2\pi$ 

amplitude: 2

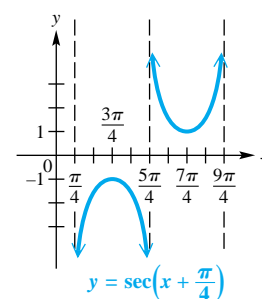
domain:  $(-\infty, \infty)$ range:  $[-2, 2]$ 

## 6.4 Graphs of the Other Circular Functions

## Cosecant and Secant Functions

**Domain:**  $\{x \mid x \neq n\pi, \text{ where } n \text{ an integer}\}$ **Range:**  $(-\infty, -1] \cup [1, \infty)$ **Period:**  $2\pi$ **Domain:**  $\{x \mid x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ an integer}\}$ **Range:**  $(-\infty, -1] \cup [1, \infty)$ **Period:**  $2\pi$ 

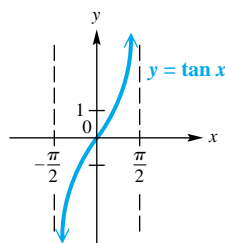
See page 577 for a summary of graphing techniques.

Graph one period of  $y = \sec(x + \frac{\pi}{4})$ .period:  $2\pi$ phase shift:  $-\frac{\pi}{4}$ domain:  $\{x \mid x \neq \frac{\pi}{4} + n\pi, \text{ where } n \text{ is an integer}\}$ range:  $(-\infty, -1] \cup [1, \infty)$ 

(continued)

## CONCEPTS

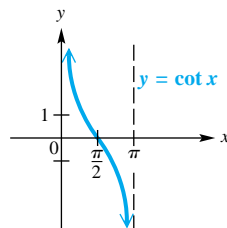
## Tangent and Cotangent Functions



**Domain:**  $\{x \mid x \neq (2n + 1)\frac{\pi}{2},$   
where  $n$  is an integer  $\}$

**Range:**  $(-\infty, \infty)$

**Period:**  $\pi$



**Domain:**  $\{x \mid x \neq n\pi,$   
where  $n$  is an integer  $\}$

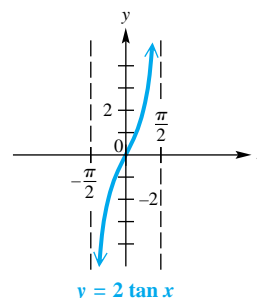
**Range:**  $(-\infty, \infty)$

**Period:**  $\pi$

See page 581 for a summary of graphing techniques.

## EXAMPLES

Graph one period of  $y = 2 \tan x$ .



period:  $\pi$

domain:  $\{x \mid x \neq (2n + 1)\frac{\pi}{2},$   
where  $n$  is an integer  $\}$

range:  $(-\infty, \infty)$

## 6.5 Harmonic Motion

## Simple Harmonic Motion

The position of a point oscillating about an equilibrium position at time  $t$  is modeled by either

$$s(t) = a \cos \omega t \quad \text{or} \quad s(t) = a \sin \omega t,$$

where  $a$  and  $\omega$  are constants, with  $\omega > 0$ . The amplitude of the motion is  $|a|$ , the period is  $\frac{2\pi}{\omega}$ , and the frequency is  $\frac{\omega}{2\pi}$ .

A spring oscillates according to

$$s(t) = -5 \cos 6t,$$

where  $t$  is in seconds and  $s(t)$  is in inches.

$$\text{amplitude} = |-5| = 5$$

$$\text{period} = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\text{frequency} = \frac{3}{\pi}$$

## Chapter 6 Review Exercises

1. (a) II (b) III (c) III (d) I

3.  $\frac{2\pi}{3}$  4.  $\frac{40\pi}{9}$  5.  $225^\circ$

6.  $-216^\circ$

1. Consider each angle in standard position having the given radian measure. In what quadrant does the terminal side lie?

(a) 3

(b) 4

(c)  $-2$

(d) 7



2. Which is larger—an angle of  $1^\circ$  or an angle of 1 radian? Discuss and justify your answer.

Convert each degree measure to radians. Leave answers as multiples of  $\pi$ .

3.  $120^\circ$

4.  $800^\circ$

Convert each radian measure to degrees.

5.  $\frac{5\pi}{4}$

6.  $-\frac{6\pi}{5}$



7. (a)  $\frac{\pi}{200}$  (b)  $3.15^\circ$ ; .055 radian  
 8. (a)  $\frac{4\pi}{3}$  in. (b)  $12\pi$  in.  
 9. 35.8 cm 10. 2263 in.<sup>2</sup>  
 11. 41 yd 12.  $\frac{1}{2}$  radian  
 13.  $\frac{1}{2}$  radian 14. 16 sq units  
 15.  $\frac{15}{32}$  sec 16. 108 radians  
 17.  $1260\pi$  m per sec  
 18.  $\frac{\pi}{36}$  radian per sec

7. **Railroad Engineering** The term *grade* has several different meanings in construction work. Some engineers use the term *grade* to represent  $\frac{1}{100}$  of a right angle and express grade as a percent. For instance, an angle of  $.9^\circ$  would be referred to as a 1% grade. (Source: Hay, W., *Railroad Engineering*, John Wiley & Sons, 1982.)

- (a) By what number should you multiply a grade to convert it to radians?  
 (b) In a rapid-transit rail system, the maximum grade allowed between two stations is 3.5%. Express this angle in degrees and radians.

8. **Concept Check** Suppose the tip of the minute hand of a clock is 2 in. from the center of the clock. For each of the following durations, determine the distance traveled by the tip of the minute hand.

- (a) 20 min (b) 3 hr

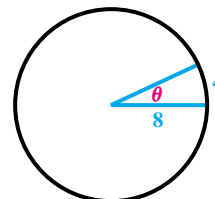


Solve each problem. Use a calculator as necessary.

9. **Arc Length** The radius of a circle is 15.2 cm. Find the length of an arc of the circle intercepted by a central angle of  $\frac{3\pi}{4}$  radians.  
 10. **Area of a Sector** A central angle of  $\frac{7\pi}{4}$  radians forms a sector of a circle. Find the area of the sector if the radius of the circle is 28.69 in.  
 11. **Height of a Tree** A tree 2000 yd away subtends an angle of  $1^\circ 10'$ . Find the height of the tree to two significant digits.  
 12. **Rotation of a Seesaw** The seesaw at a playground is 12 ft long. Through what angle does the board rotate when a child rises 3 ft along the circular arc?

Consider the figure here for Exercises 13 and 14.

13. What is the measure of  $\theta$  in radians?  
 14. What is the area of the sector?



Solve each problem.

15. Find the time  $t$  if  $\theta = \frac{5\pi}{12}$  radians and  $\omega = \frac{8\pi}{9}$  radians per sec.  
 16. Find angle  $\theta$  if  $t = 12$  sec and  $\omega = 9$  radians per sec.  
 17. **Linear Speed of a Flywheel** Find the linear speed of a point on the edge of a flywheel of radius 7 m if the flywheel is rotating 90 times per sec.  
 18. **Angular Speed of a Ferris Wheel** A Ferris wheel has radius 25 ft. If it takes 30 sec for the wheel to turn  $\frac{5\pi}{6}$  radians, what is the angular speed of the wheel?



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19.  $A = \pi r^2$ , the formula for the area of a circle 20.  $-\frac{1}{2}$

21.  $-\sqrt{3}$  22. 2

23. .9703 24.  $-11.4266$

25. .5148 26.  $\frac{2\pi}{3}$  27.  $\frac{7\pi}{6}$

28. (a) 0; The face of the moon

is not visible. (b)  $\frac{1}{2}$ ; Half the face of the moon is visible.

(c) 1; The face of the moon is

completely visible. (d)  $\frac{1}{2}$ ; Half the face of the moon is visible.

19. **Concept Check** Consider the area-of-a-sector formula  $A = \frac{1}{2}r^2\theta$ . What well-known formula corresponds to the special case  $\theta = 2\pi$ ?

Find the exact function value. Do not use a calculator.

20.  $\cos \frac{2\pi}{3}$

21.  $\tan\left(-\frac{7\pi}{3}\right)$

22.  $\csc\left(-\frac{11\pi}{6}\right)$

Use a calculator to find an approximation for each circular function value. Be sure your calculator is set in radian mode.

23.  $\cos(-.2443)$

24.  $\cot 3.0543$

25. Approximate the value of  $s$  in the interval  $\left[0, \frac{\pi}{2}\right]$  if  $\sin s = .4924$ .

Find the exact value of  $s$  in the given interval that has the given circular function value. Do not use a calculator.

26.  $\left[\frac{\pi}{2}, \pi\right]$ ;  $\tan s = -\sqrt{3}$

27.  $\left[\pi, \frac{3\pi}{2}\right]$ ;  $\sec s = -\frac{2\sqrt{3}}{3}$

(Modeling) Solve each problem.

28. **Phase Angle of the Moon** Because the moon orbits Earth, we observe different phases of the moon during the period of a month. In the figure,  $t$  is called the *phase angle*.



The phase  $F$  of the moon is computed by

$$F(t) = \frac{1}{2}(1 - \cos t),$$

and gives the fraction of the moon's face that is illuminated by the sun. (Source: Duffet-Smith, P., *Practical Astronomy with Your Calculator*, Cambridge University Press, 1988.) Evaluate each expression and interpret the result.

(a)  $F(0)$  (b)  $F\left(\frac{\pi}{2}\right)$  (c)  $F(\pi)$  (d)  $F\left(\frac{3\pi}{2}\right)$

29. **Atmospheric Effect on Sunlight** The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of  $\csc \theta$ , where  $\theta$  is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reaches Earth's surface. See the figure at the top of the next page. (Source: Winter, C., R. Sizmann, and Vant-Hunt (Editors), *Solar Power Plants*, Springer-Verlag, 1991.)

29. (b)  $\frac{\pi}{6}$  (c) less ultraviolet

light when  $\theta = \frac{\pi}{3}$  30. B

31. D 32. 2;  $2\pi$ ; none; none

33. not applicable;  $\frac{\pi}{3}$ ; none;

none 34.  $\frac{1}{2}$ ;  $\frac{2\pi}{3}$ ; none; none

35. 2;  $\frac{2\pi}{5}$ ; none; none 36. 2;

$8\pi$ ; 1 up; none 37.  $\frac{1}{4}$ ;  $3\pi$ ;

3 up; none 38. 3;  $2\pi$ ; none;

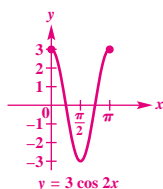
$\frac{\pi}{2}$  to the left 39. 1;  $2\pi$ ; none;

$\frac{3\pi}{4}$  to the right 40. not

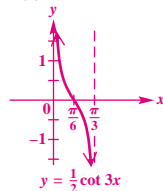
applicable;  $\pi$ ; none;  $\frac{\pi}{8}$  to the

right 41. tangent 42. sine

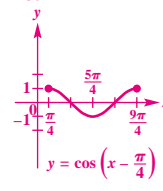
43. cosine 44.



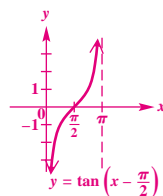
45.



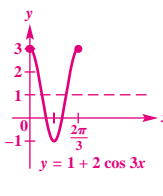
46.



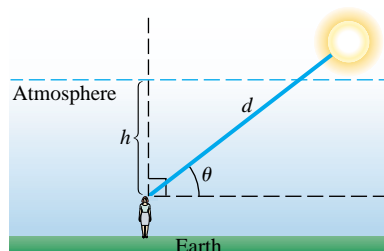
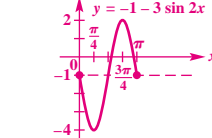
47.



48.



49.



(a) Verify that  $d = h \csc \theta$ .

(b) Determine  $\theta$  when  $d = 2h$ .

(c) The atmosphere filters out the ultraviolet light that causes skin to burn. Compare the difference between sunbathing when  $\theta = \frac{\pi}{2}$  and when  $\theta = \frac{\pi}{3}$ . Which measure gives less ultraviolet light?

30. **Concept Check** Which one of the following is true about the graph of  $y = 4 \sin 2x$ ?

A. It has amplitude 2 and period  $\frac{\pi}{2}$ .

B. It has amplitude 4 and period  $\pi$ .

C. Its range is  $[0, 4]$ .

D. Its range is  $[-4, 0]$ .

31. **Concept Check** Which one of the following is false about the graph of  $y = -3 \cos \frac{1}{2}x$ ?

A. Its range is  $[-3, 3]$ .

B. Its domain is  $(-\infty, \infty)$ .

C. Its amplitude is 3, and its period is  $4\pi$ .

D. Its amplitude is 3, and its period is  $\pi$ .

For each function, give the amplitude, period, vertical translation, and phase shift, as applicable.

32.  $y = 2 \sin x$

33.  $y = \tan 3x$

34.  $y = -\frac{1}{2} \cos 3x$

35.  $y = 2 \sin 5x$

36.  $y = 1 + 2 \sin \frac{1}{4}x$

37.  $y = 3 - \frac{1}{4} \cos \frac{2}{3}x$

38.  $y = 3 \cos\left(x + \frac{\pi}{2}\right)$

39.  $y = -\sin\left(x - \frac{3\pi}{4}\right)$

40.  $y = \frac{1}{2} \csc\left(2x - \frac{\pi}{4}\right)$

**Concept Check** Identify the circular function that satisfies each description.

41. period is  $\pi$ ,  $x$ -intercepts are of the form  $n\pi$ , where  $n$  is an integer

42. period is  $2\pi$ , graph passes through the origin

43. period is  $2\pi$ , graph passes through the point  $(\frac{\pi}{2}, 0)$

Graph each function over a one-period interval.

44.  $y = 3 \cos 2x$

45.  $y = \frac{1}{2} \cot 3x$

46.  $y = \cos\left(x - \frac{\pi}{4}\right)$

47.  $y = \tan\left(x - \frac{\pi}{2}\right)$

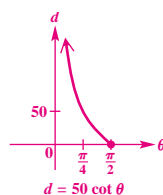
48.  $y = 1 + 2 \cos 3x$

49.  $y = -1 - 3 \sin 2x$

50. Explain how by observing the graphs of  $y = \sin x$  and  $y = \cos x$  on the same axes, one can see that for exactly two  $x$ -values in  $[0, 2\pi)$ ,  $\sin x = \cos x$ . What are the two  $x$ -values?

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51. (b)

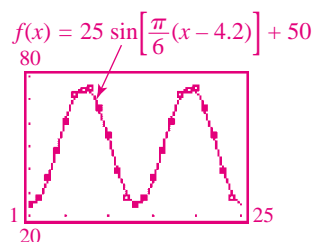


52. (a) 12.3 hr (b) 1.2 ft  
(c) 1.56 ft 53. (a)  $30^\circ$   
(b)  $60^\circ$  (c)  $75^\circ$  (d)  $86^\circ$   
(e)  $86^\circ$  (f)  $60^\circ$

54. (a) See the graph in part (d).  
(b)

$$f(x) = 25 \sin \left[ \frac{\pi}{6}(x - 4.2) \right] + 50$$

(d) The function gives an excellent model for the data.



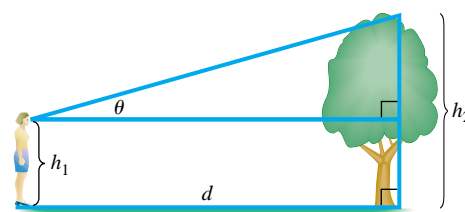
(e) 

```
SinReg
y=a*sin(bx+c)+d
a=25.77
b=.52
c=-2.19
d=50.57
```

TI-83 Plus fixed to the nearest hundredth.

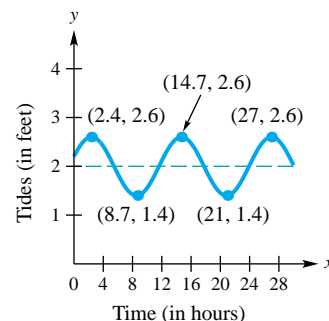
Solve each problem.

51. **Viewing Angle to an Object** Let a person  $h_1$  feet tall stand  $d$  feet from an object  $h_2$  feet tall, where  $h_2 > h_1$ . Let  $\theta$  be the angle of elevation to the top of the object. See the figure.



- (a) Show that  $d = (h_2 - h_1) \cot \theta$ .  
(b) Let  $h_2 = 55$  and  $h_1 = 5$ . Graph  $d$  for the interval  $0 < \theta \leq \frac{\pi}{2}$ .

52. **(Modeling) Tides** The figure shows a function  $f$  that models the tides in feet at Clearwater Beach, Florida,  $x$  hours after midnight starting on August 26, 1998. (Source: Pentcheff, D., *WWW Tide and Current Predictor*.)



- (a) Find the time between high tides.  
(b) What is the difference in water levels between high tide and low tide?  
(c) The tides can be modeled by

$$f(x) = .6 \cos[.511(x - 2.4)] + 2.$$

Estimate the tides when  $x = 10$ .

53. **(Modeling) Maximum Temperatures** The maximum afternoon temperature in a given city might be modeled by

$$t = 60 - 30 \cos \frac{x\pi}{6},$$

where  $t$  represents the maximum afternoon temperature in month  $x$ , with  $x = 0$  representing January,  $x = 1$  representing February, and so on. Find the maximum afternoon temperature for each month.

- (a) January (b) April (c) May  
(d) June (e) August (f) October

## 54. (Modeling) Average Monthly Temperature

The average monthly temperature (in  $^\circ\text{F}$ ) in Chicago, Illinois, is shown in the table.

- (a) Plot the average monthly temperature over a two-year period. Let  $x = 1$  correspond to January of the first year.  
(b) Determine a model function of the form  $f(x) = a \sin b(x - d) + c$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants.  
(c) Explain the significance of each constant.  
(d) Graph  $f$  together with the data on the same coordinate axes. How well does  $f$  model the data?  
(e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

Month	$^\circ\text{F}$	Month	$^\circ\text{F}$
Jan	25	July	74
Feb	28	Aug	75
Mar	36	Sept	66
Apr	48	Oct	55
May	61	Nov	39
June	72	Dec	28

Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology*, 4th Edition, Charles E. Merrill Publishing Co., 1983.

55. **(Modeling) Pollution Trends** The amount of pollution in the air fluctuates with the seasons. It is lower after heavy spring rains and higher after periods of little rain. In addition to this seasonal fluctuation, the long-term trend is upward. An idealized graph of this situation is shown in the figure on the next page.

55. (a) 100 (b) 258 (c) 122  
(d) 296 56. amplitude: 3;

period:  $\pi$ ; frequency:  $\frac{1}{\pi}$

57. amplitude: 4; period: 2;

frequency:  $\frac{1}{2}$  58. The period is

the time to complete one cycle.

The amplitude is the maximum distance (on either side) from the initial point. 59. The frequency is the number of cycles in one unit of time;  $-4$ ;  $0$ ;  $-2\sqrt{2}$

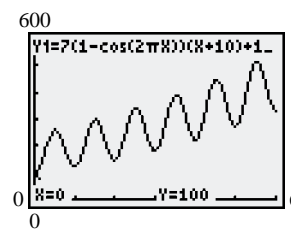
55. (continued) Circular functions can be used to model the fluctuating part of the pollution levels, and exponential functions can be used to model long-term growth. The pollution level in a certain area might be given by

$$y = 7(1 - \cos 2\pi x)(x + 10) + 100e^{-2x},$$

where  $x$  is the time in years, with  $x = 0$  representing

January 1 of the base year. July 1 of the same year would be represented by  $x = .5$ , October 1 of the following year would be represented by  $x = 1.75$ , and so on. Find the pollution levels on each date.

- (a) January 1, base year (b) July 1, base year  
(c) January 1, following year (d) July 1, following year



An object in simple harmonic motion has position function  $s$  inches from an initial point, where  $t$  is the time in seconds. Find the amplitude, period, and frequency.

56.  $s(t) = 3 \cos 2t$

57.  $s(t) = 4 \sin \pi t$

58. In Exercise 56, what does the period represent? What does the amplitude represent?  
59. In Exercise 57, what does the frequency represent? Find the position of the object from the initial point at 1.5 sec, 2 sec, and 3.25 sec.

## Chapter 6 Test

1.  $\frac{2\pi}{3}$  2.  $-\frac{\pi}{4}$  3. .09  
4.  $135^\circ$  5.  $-210^\circ$  6.  $229.18^\circ$   
7. (a)  $\frac{4}{3}$  (b)  $15,000 \text{ cm}^2$   
8. 2 radians 9. (a)  $\frac{\pi}{3}$  radians  
(b)  $\frac{10\pi}{3} \text{ cm}$  (c)  $\frac{5\pi}{9} \text{ cm per sec}$

Convert each degree measure to radians.

1.  $120^\circ$  2.  $-45^\circ$  3.  $5^\circ$  (to the nearest hundredth)

Convert each radian measure to degrees.

4.  $\frac{3\pi}{4}$  5.  $-\frac{7\pi}{6}$  6. 4 (to the nearest hundredth)

7. A central angle of a circle with radius 150 cm cuts off an arc of 200 cm. Find each measure.

- (a) the radian measure of the angle  
(b) the area of a sector with that central angle

8. **Rotation of Gas Gauge Arrow** The arrow on a car's gasoline gauge is  $\frac{1}{2}$  in. long. See the figure. Through what angle does the arrow rotate when it moves 1 in. on the gauge?

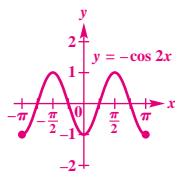


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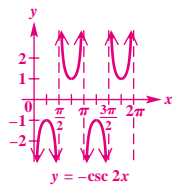
10.  $\frac{\sqrt{2}}{2}$  11.  $-\frac{\sqrt{3}}{2}$   
 12. undefined 13.  $-2$   
 14. (a) .9716 (b)  $\frac{\pi}{3}$   
 15. (a)  $\pi$  (b) 6 (c)  $[-3, 9]$   
 (d)  $-3$  (e)  $\frac{\pi}{4}$  to the left

(that is,  $-\frac{\pi}{4}$ )

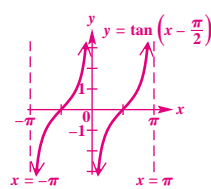
16.



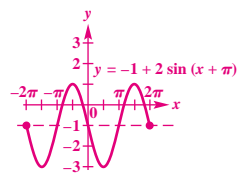
17.



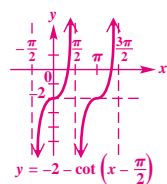
18.



19.

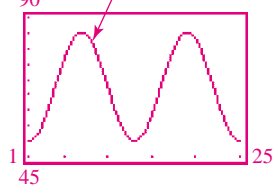


20.



21. (a)

$$f(x) = 17.5 \sin\left[\frac{\pi}{6}(x - 4)\right] + 67.5$$



- (b) 17.5; 12; 4 to the right; 67.5 up  
 (c) approximately 52°F  
 (d) 50°F in January; 85°F in July  
 (e) approximately 67.5°; This is the vertical translation.

22. (a) 4 in. (b) after  $\frac{1}{8}$  sec

(c) 4 cycles per sec;  $\frac{1}{4}$  sec

9. **Angular and Linear Speed of a Point** Suppose that point  $P$  is on a circle with radius 10 cm, and ray  $OP$  is rotating with angular speed  $\frac{\pi}{18}$  radian per sec.

- (a) Find the angle generated by  $P$  in 6 sec.  
 (b) Find the distance traveled by  $P$  along the circle in 6 sec.  
 (c) Find the linear speed of  $P$ .

Find each circular function value.

10.  $\sin \frac{3\pi}{4}$

11.  $\cos\left(-\frac{7\pi}{6}\right)$

12.  $\tan \frac{3\pi}{2}$

13.  $\sec \frac{2\pi}{3}$

14. (a) Use a calculator to approximate  $s$  in the interval  $[0, \frac{\pi}{2}]$ , if  $\sin s = .8258$ .  
 (b) Find the exact value of  $s$  in the interval  $[0, \frac{\pi}{2}]$ , if  $\cos s = \frac{1}{2}$ .  
 15. Consider the function defined by  $y = 3 - 6 \sin(2x + \frac{\pi}{2})$ .  
 (a) What is its period?  
 (b) What is the amplitude of its graph?  
 (c) What is its range?  
 (d) What is the  $y$ -intercept of its graph?  
 (e) What is its phase shift?

Graph each function over a two-period interval. Identify asymptotes when applicable.

16.  $y = -\cos 2x$

17.  $y = -\csc 2x$

18.  $y = \tan\left(x - \frac{\pi}{2}\right)$

19.  $y = -1 + 2 \sin(x + \pi)$

20.  $y = -2 - \cot\left(x - \frac{\pi}{2}\right)$

21. **(Modeling) Average Monthly Temperature** The average monthly temperature (in °F) in Austin, Texas, can be modeled using the circular function defined by

$$f(x) = 17.5 \sin\left[\frac{\pi}{6}(x - 4)\right] + 67.5,$$

where  $x$  is the month and  $x = 1$  corresponds to January. (Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology*, 4th Edition, Charles E. Merrill Publishing Co., 1983.)



- (a) Graph  $f$  in the window  $[1, 25]$  by  $[45, 90]$ .  
 (b) Determine the amplitude, period, phase shift, and vertical translation of  $f$ .  
 (c) What is the average monthly temperature for the month of December?  
 (d) Determine the maximum and minimum average monthly temperatures and the months when they occur.  
 (e) What would be an approximation for the average yearly temperature in Austin? How is this related to the vertical translation of the sine function in the formula for  $f$ ?

22. **(Modeling) Spring** The height of a weight attached to a spring is

$$s(t) = -4 \cos 8\pi t$$

inches after  $t$  seconds.

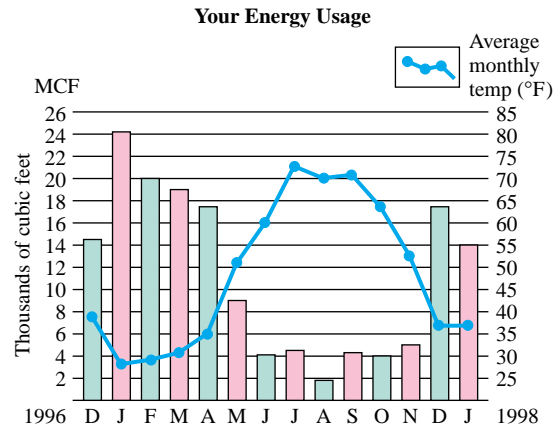
- (a) Find the maximum height that the weight rises above the equilibrium position of  $y = 0$ .  
 (b) When does the weight first reach its maximum height, if  $t \geq 0$ ?  
 (c) What are the frequency and period?

## Chapter 6 Quantitative Reasoning



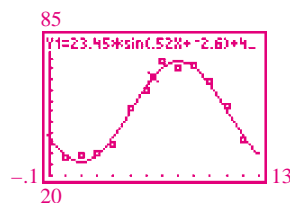
### Does the fact that average monthly temperatures are periodic affect your utility bills?

In an article entitled "I Found Sinusoids in My Gas Bill" (*Mathematics Teacher*, January 2000), Cathy G. Schloemer presents the following graph that accompanied her gas bill.



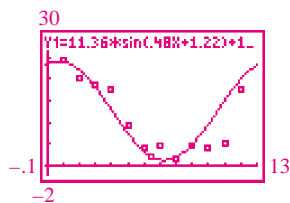
1. SinReg  
 $y = a \sin(bx + c) + d$   
 $a = 23.45$   
 $b = .52$   
 $c = -2.60$   
 $d = 49.94$

TI-83 Plus fixed to the nearest hundredth



2. SinReg  
 $y = a \sin(bx + c) + d$   
 $a = 11.36$   
 $b = .48$   
 $c = 1.22$   
 $d = 12.69$

TI-83 Plus fixed to the nearest hundredth



Notice that two sinusoids are suggested here: one for the behavior of the average monthly temperature and another for gas use in MCF (thousands of cubic feet).

- If January 1997 is represented by  $x = 1$ , the data of estimated ordered pairs (month, temperature) is given in the list shown on the two graphing calculator screens below.

L1	L2	L3	1
1	23		
2	20		
3	19		
4	17		
5	16		
6	14		
7	12		
8	10		
9	8		
10	6		
11	4		
12	2		
L1(1)=1			

L1	L2	L3	1
1	23		
2	20		
3	19		
4	17		
5	16		
6	14		
7	12		
8	10		
9	8		
10	6		
11	4		
12	2		
L1(13)=			

Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

- If January 1997 is again represented by  $x = 1$ , the data of estimated ordered pairs (month, gas use in MCF) is given in the list shown on the two graphing calculator screens below.

L1	L2	L3	1
1	24.2		
2	20		
3	18.8		
4	17.5		
5	16		
6	14.2		
7	12		
8	10		
9	8		
10	6		
11	4		
12	2		
L1(1)=1			

L1	L2	L3	1
1	4.8		
2	1.8		
3	4.8		
4	4		
5	5		
6	17.5		
7			
8			
9			
10			
11			
12			
L1(13)=			

Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

- Answer the question posed at the beginning of this exercise, in the form of a short paragraph.