GRAPHS OF THE CIRCULAR FUNCTIONS

1. *GRAPHS OF THE SINE AND COSINE FUNCTIONS*

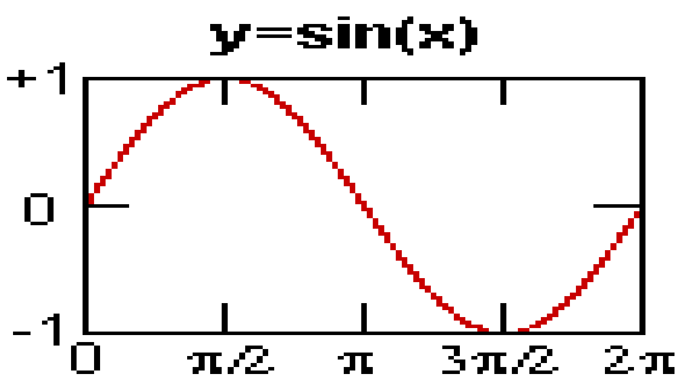
- PERIODIC FUNCTION

> A **period function** is a function f such that $f(x) = f(x+np)$, for every real number *x* in the domain of f, every integer *n*, and some positive real number *p*. The smallest possible value of *p* is the **period** of the function.

- GRAPH OF THE SINE FUNCTION
  - Recall that for a real number *s*, the point on the unit circle corresponding to *s* has coordinates $(\cos s, \sin s)$.

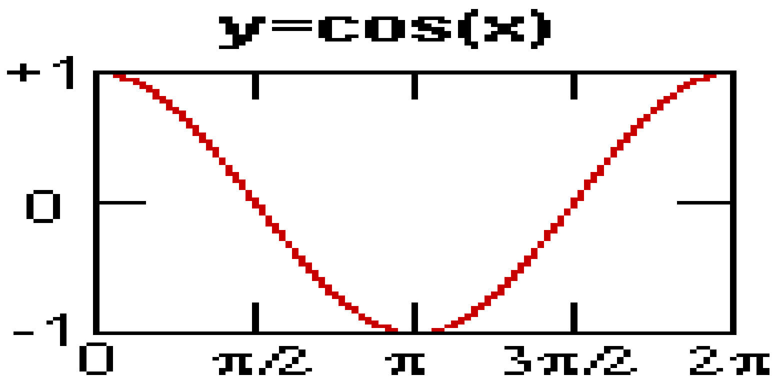
As *s* increases from	sin *s*	cos *s*
0 to $\frac{\pi}{2}$	Increases from 0 to 1	Decreases from 1 to 0
$\frac{\pi}{2}$ to π	Decreases from 1 to 0	Decreases from 0 to -1
π to $\frac{3\pi}{2}$	Decreases from 0 to -1	Increases from -1 to 0
$\frac{3\pi}{2}$ to 2π	Increases from -1 to 0	Increases from 0 to 1

x	*y*	*(x,y)*
0	0	$(0,0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6},\frac{1}{2}\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\pi}{3},\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2},1\right)$
π	0	$(\pi,0)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2},-1\right)$
2π	0	$(2\pi,0)$



- The graph is continuous over the entire domain, $(-\infty,\infty)$
- Its *x*-intercepts are of the form $n\pi, \ni n \in Z$.
- Its period is 2π.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all *x* in the domain, $\sin(-x) = -\sin(x)$.

x	y	(x,y)
0	0	$(0,0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\pi}{6},\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3},\frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2},0\right)$
π	-1	$(\pi,-1)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2},0\right)$
2π	1	$(2\pi,1)$



- The graph is continuous over the entire domain, $(-\infty,\infty)$
- Its x-intercepts are of the form $(2n+1)\frac{\pi}{2}, \ni n \in \mathrm{Z}.$
- Its period is 2π.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $\cos(-x)=\cos(x)$.

- AMPLITUDE

> The graph of $y=a\sin x$ or $y=a\cos x$, with $a\neq 0$, will have the same shape as the graph of $y=\sin x$ or $y=\cos x$, respectively, except with range $\left[-|a|,|a|\right]$. The amplitude is $|a|$.

- PERIOD

For $b>0$, the graph of $y=\sin bx$ will resemble that of $y=\sin x$, but with period $\frac{2\pi}{b}$. Also, the graph of $y=\cos bx$ will resemble that of $y=\cos x$, but with period $\frac{2\pi}{b}$.

- GUIDELINES FOR SKETCHING GRAPHS OF THE SINE AND COSINE FUNCTIONS

To graph $y = a\sin bx$ or $y = a\cos bx$, with $b > 0$, follow these steps.

1. Find the period, $\frac{2\pi}{b}$. Start at 0 on the *x*-axis, and lay off a distance of $\frac{2\pi}{b}$.
2. Divide the interval into 4 equal parts.
3. Evaluate the function for each of the five angle from the *x*-axis. This will give the *x*-intercepts, maximum points, and minimum points.
4. Plot the points found in (3), and join them with a sinusoidal curve having amplitude $|a|$.
5. Draw the graph over additional periods, to the right and to the left, as needed.

2. TRANSLATIONS OF THE GRAPHS OF THE SINE AND COSINE

- HORIZONTAL TRANSLATIONS

  - The graph of the function defined by $y = f(x-d)$ is translated horizontally compared to the graph of $f(x)$. The translation is *d* units to the right if $d > 0$, or $|d|$ units to the left if $d < 0$. With circular functions, a horizontal translation is called a **phase shift**, and $x-d$ is called the **argument**.

  - How to graph:
    - Method 1: Consider $y = \sin\left(x + \frac{\pi}{4}\right)$

      - Solve the inequality:

$$0 \le x + \frac{\pi}{4} \le 2\pi$$

$$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$$

- Divide the resulting interval into 4 equal parts to get the following *x* values:

$$-\frac{\pi}{4},\quad \frac{\pi}{4},\quad \frac{3\pi}{4},\quad \frac{5\pi}{4},\quad \frac{7\pi}{4}$$

x	$x+\frac{\pi}{4}$	$\sin\left(x+\frac{\pi}{4}\right)$	**(x,y)**
$-\frac{\pi}{4}$	0	0	$\left(-\frac{\pi}{4},0\right)$
$\frac{\pi}{4}$	$\frac{\pi}{2}$	1	$\left(\frac{\pi}{4},1\right)$
$\frac{3\pi}{4}$	π	0	$\left(\frac{3\pi}{4},0\right)$
$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	-1	$\left(\frac{5\pi}{4},-1\right)$
$\frac{7\pi}{4}$	2π	0	$\left(\frac{7\pi}{4},0\right)$

- Method 2: Consider $y=\sin\left(x+\frac{\pi}{4}\right)$
  - Graph $y=\sin(x)$ and then graph it again (using a different type of line or color) shifted to the left $\frac{\pi}{4}$.

- VERTICAL TRANSLATIONS
  - The graph of the function defined by $y=c+f(x)$ is translated vertically compared to the graph of $f(x)$. The translation is *c* units up if $c>0$, or $|c|$ units down if $c<0$.
  - My suggestion is to use a combination of methods (1) and (2) to graph functions of the type $y=c+f(x-d)$

- FURTHER GUIDELINES FOR SKETCHING GRAPHS OF SINE AND COSINE FUNCTIONS IN THE FORM OF $y = c + a \sin b(x-d)$ or $y = c + a \cos b(x-d)$

METHOD 1:

STEP 1 Find an interval whose length is one period $\frac{2\pi}{b}$ by solving the three-part inequality $0 \le b(x-d) \le 2\pi.$

STEP 2 Divide the interval into four equal parts.

STEP 3 Evaluate the function for each of the five *x* values resulting from step 2. The points will be maximum points, minimum points, and points that intersect the line $y = c$ ("middle points of the wave").

STEP 4 Plot the points found in step 3, and join them with a sinusoidal curve having amplitude $|a|$.

STEP 5 Draw the graph over additional periods, to the right and to the left, as needed.

METHOD 2:

First graph the basic circular function. The amplitude of the function is $|a|$, and the period of the function is $\frac{2\pi}{b}$. Then use translations to graph the desired function. The vertical translation is if $c > 0,$ or $|c|$ units down if $c < 0.$ The translation is *d* units to the right if $d > 0,$ or $|d|$ units to the left if $d < 0.$ The horizontal translation is *d* units to the right if $d > 0,$ or $|d|$ units to the left if $d < 0.$

- EXAMPLE: Graph $y = 1 + \frac{2}{3}\cos\frac{1}{2}x$

Step 1:

$$0 \le b(x-d) \le 2\pi$$
$$0 \le \frac{1}{2}(x-0) \le 2\pi$$
$$0 \le \frac{1}{2}x \le 2\pi$$
$$0 \le x \le 4\pi$$

Step 2: $0, \quad \pi, \quad 2\pi, \quad 3\pi, \quad 4\pi$

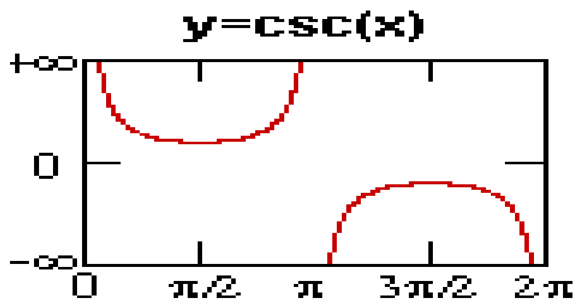
Step 3:

x	$\frac{1}{2}x$	$1+\frac{2}{3}\cos\frac{1}{2}x$	**(*x*,*y*)**
0	0	$\frac{5}{3}$	$\left(0,\frac{5}{3}\right)$
π	$\frac{\pi}{2}$	1	$(\pi,1)$
2π	π	$\frac{1}{3}$	$\left(2\pi,\frac{1}{3}\right)$
3π	$\frac{3\pi}{2}$	1	$(3\pi,1)$
4π	2π	$\frac{5}{3}$	$\left(4\pi,\frac{5}{3}\right)$

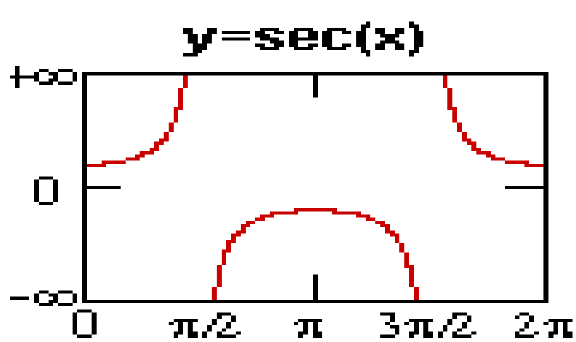
Now you try making the graph.

a. *GRAPHS OF THE OTHER CIRCULAR FUNCTIONS*

- GRAPHS OF THE COSECANT AND SECANT FUNCTIONS
  - Cosecant
    - Since cosecant values are reciprocals of the corresponding sine values, the period of the function $y = \csc x$ is 2π.
    - When $\sin x = 1$ $\csc x = 1$, and when $0 < \sin x < 1$, $\csc x > 1$.
    - Similarly, when $-1 < \sin x < 0$, $\csc x < -1$.
    - As $|x| \to 0$, $|\sin x| \to 0$, and $|\csc x| \to \infty$. Therefore the vertical lines at $x = n\pi$ are all vertical asymptotes.



  - Secant
    - Since secant values are reciprocals of the corresponding cosine values, the period of the function $y = \sec x$ is 2π.
    - When $\cos x = 1$ $\sec x = 1$, and when $0 < \cos x < 1$, $\sec x > 1$.
    - Similarly, when $-1 < \cos x < 0$, $\sec x < -1$.
    - As $|x| \to \frac{\pi}{2}$, $|\cos x| \to 0$, and $|\sec x| \to \infty$. Therefore the vertical lines at $x = \frac{\pi}{2} + n\pi$ are all vertical asymptotes.



- GUIDELINES FOR SKETCHING GRAPHS OF COSECANT AND SECANT FUNCTIONS

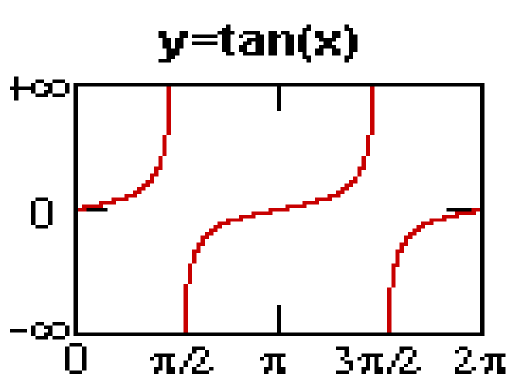
  To graph $y = a\csc bx$ or $y = a\sec bx$, with $b > 0$, follow these steps.

  Step 1 Graph the corresponding reciprocal function as a guide, using a dashed curve.

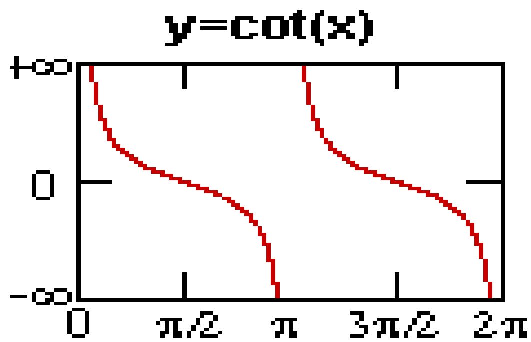
  Step 2 Sketch the vertical asymptotes.

  Step 3 Sketch the graph of the desired function by drawing the typical U-shaped branches between the adjacent asymptotes. The branches will be above the graph of the guide function when the guide function values are positive and below the graph of the guide function when the guide function values are negative.

- *GRAPHS OF THE TANGENT AND COTANGENT FUNCTIONS*
  - *Tangent*
    - Period is π.
    - Since $\tan x = \dfrac{\sin x}{\cos x}$, tangent values are zero when the sine values are zero and undefined when the cosine values are zero.
    - As *x*-values go from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$, tangent values go from $-\infty$ to ∞, and **increase throughout the interval**.



- *Cotangent*
  - Period is π.
  - Since $\cot x = \dfrac{\cos x}{\sin x}$, tangent values are zero when the cosine values are zero and undefined when the sine values are zero.
  - As *x*-values go from 0 to π, tangent values go from $-\infty$ to ∞, and **decrease throughout the interval**.



- GUIDELINES FOR SKETCHING GRAPHS OF TANGENT AND COTANGENT FUNCTIONS

  To graph $y = a\tan bx$ or $y = a\cot bx$, with $b > 0$, follow these steps:

Step 1 Determine the period, $\dfrac{\pi}{b}$. To locate two adjacent vertical asymptotes, solve the following equations for x:

For $y = a\tan x$: $bx = -\dfrac{\pi}{2}$ and $bx = \dfrac{\pi}{2}$.

For $y = a\cot x$: $bx = 0$ and $bx = \pi$.

Step 2 Sketch the two vertical asymptotes found in Step 1.

Step 3 Divide the interval formed by the vertical asymptotes into four equal parts.

Step 4 Evaluate the function for the first-quarter point, midpoint, and third-quarter point, using the *x*-values found in Step 3.

Step 5 Join the points with a smooth curve, approaching the vertical asymptotes. Indicate additional asymptotes and periods of the graph as necessary.

b. HARMONIC MOTION

- SIMPLE HARMONIC MOTION
  - A spring with a weight attached to its free end is in equilibrium (or rest) position.
  - If the weight is pulled down *a* units and released, the spring's elasticity causes the weight to rise *a* units $(a > 0)$ above the equilibrium position, and then oscillate about the equilibrium position. This motion can be described mathematically by a sinusoid, if friction is neglected.

The position of a point oscillating about an equilibrium position at time *t* is modeled by either $s(t) = a\cos\omega t \quad \text{or} \quad s(t) = a\sin\omega t,$ where a and ω are constants, with $\omega > 0.$ The amplitude of the motion is $\lvert a \rvert,$ the period is $\dfrac{2\pi}{\omega},$ and the frequency is $\dfrac{\omega}{2\pi}.$

  - Example: An object is attached to a coiled spring. It is pulled down a distance of 6 units from its equilibrium position, and then released. The time for one complete oscillation is 4 seconds.

a) Give an equation that models the position of the object at time $t.$
Solution: When the object is released at $t = 0,$ the distance of the object from the equilibrium position is 6 units below equilibrium. If $s(t)$ is to model the motion, then $s(0)$ must equal $-6.$ We use $s(t) = a\cos\omega t,$ with $a = -6.$ We choose the cosine function since $\cos\omega(0) = \cos 0 = 1,$ and $-6 \cdot 1 = -6.$ Had we chosen the sine function, a phase shift would have been required. The period is 4, so $\dfrac{2\pi}{\omega} = 4 \Rightarrow 4\omega = 2\pi \Rightarrow \omega = \dfrac{\pi}{2}.$ So we have $s(t) = -6\cos\dfrac{\pi}{2}(t).$

b) Determine the position at $t = 1.25$ seconds.

Solution:

$s(1.25) = -6\cos\frac{\pi}{2}\left(1.25\right) = -6\cos\frac{5\pi}{8} = -6\left(-0.382683\right) \approx 2.30$

units.

c) Find the frequency.

The frequency is $\frac{\omega}{2\pi} = \frac{\pi/2}{2\pi} = \frac{\pi}{2}\cdot\frac{1}{2\pi} = \frac{1}{4}$.

- DAMPED OSCILLATORY MOTION

  - Previously, we did not consider the effect of friction. Friction causes the amplitude of the motion to diminish gradually until the weight comes to rest. This is referred to as the motion has been **damped** by the force of friction.

  - A typical example of **damped oscillatory motion** is provided by the function defined by $s(t) = e^{-t}\sin t$.