MATHS METHODS

QUADRATICS REVIEW

LAWS OF EXPANSION

A reminder of some of the laws of expansion, which in reverse are a quick reference for rules of factorisation

$$
\begin{aligned}
a(b+c) &= ab+ac \\
(a+b)(c+d) &= ac+ad+bc+bd \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a-b)^2 &= a^2-2ab+b^2 \\
(a+b)(a-b) &= a^2-b^2
\end{aligned}
$$

For example:

a) $3x^2-12x=3x(x-4x)$
b) $x^2+5x+6=(x+2)(x+3)$
c) $x^2+50x+25=(x+5)^2$
d) $x^2-32x+16=(x-4)^2$
e) $x^2-9=(x-3)(x+3)$

Exercise 4A Q6ace,7acegi

FACTORISING

- When factorising a quadratic expression, we end up with the product of two linear expressions
- There are four main methods of factorising:

COMMON FACTOR

$$ab+ac=a(b+c)$$

- Remove the highest common factor of the expression

For example:

a) $2x^2+3x=x(2x+3)$
b) $8x^2-6x(3-x)=8x^2-18x+6x^2=14x^2-18x=2x(7x-9)$

DIFFERENCE OF TWO SQUARES (DOTS)

$$a^2-b^2=(a+b)(a-b)$$

- All terms of the expression must be 'square' numbers

For example:

a) $4x^2+9=(2x+3)(2x-3)$
b) $100-4x^2=(10+2x)(10-2x)$

PERFECT SQUARES

$$x^2 + 2ax + a^2 = (x+a)^2$$
$$x^2 - 2ax + a^2 = (x-a)^2$$

- Where you can see the last term is a square number, and middle term is two times the first and last

For example:
- **a)** $x^2 + 10x + 25 = (x+5)^2$
- **b)** $2x^2 - 28x - 98 = 2(x^2 - 14x - 49) = 2(x-7)^2$

SUM AND PRODUCT FACTORISATION

$$x^2 + \underbrace{(p+q)}_{\text{sum of } p \text{ and } q}x + \underbrace{pq}_{\text{product of } p \text{ and } q} = (x+p)(x+q)$$

- When none of the previous methods apply, we are looking for the two numbers which add to give the middle value, and multiply to give the last value

For example:
- **a)** $x^2 + 2x - 63 = (x+9)(x-7)$
- **b)** $-3x^2 - 30x - 27 = -3(x^2 + 10x + 9) = -3(x+1)(x+9)$

IN GERNERAL…

Step 1: If the expression has a **common factor**, take it out.

Step 2: Look for the **difference of two squares**: $x^2 - a^2 = (x+a)(x-a)$

Step 3: Look for a **perfect square** factorisation: $x^2 + 2ax + a^2 = (x+a)^2$ or $x^2 - 2ax + a^2 = (x-a)^2$

Step 4: Look for the **sum and product** type: $x^2 + (p+q)x + pq = (x+p)(x+q)$

Exercise 4B Q 2adgj, 3ad, 4adg, 5adgjm

SOLUTIONS TO QUADRATICS

- When asked to find the solution/s to a quadratic equation, we are looking to solve for *x*
- In most instances we use the **NULL FACTOR LAW**

NULL FACTOR LAW

- For a factorised expression, the value/s of *x* which make each factor equal to zero

For example:
- **a)** For $(x+9)(x-7), x = -9 \text{ or } x = 7$
- **b)** For $-3(x+1)(x+9), x = -1 \text{ or } x = -9$
- **c)** For $2x(7x-9), x = 0 \text{ or } x = \frac{9}{7}$
- **d)** For $(x-7)^2, x = 7$

SOLVING QUADRATICS IN GENERAL…

Step 1: If necessary, rearrange the equation so one side is **zero**.
Step 2: **Fully factorise** the other side (usually the LHS).
Step 3: Use the **Null Factor law**.
Step 4: **Solve** the resulting linear equations.
Step 5: **Check** at least one of your solutions.

Exercise 4C Q3adgjmp, 5, 6, 9, 12

GRAPHING QUADRATICS

- A quadratic function is one which can be written in the form $f(x)=ax^2+bx+c$, where *a, b* and *c* are constants and $a \neq 0$
- This form is known as **polynomial form**
- The **axis of symmetry** for the general quadratic $y=ax^2+bx+c$ occurs at

$$\boxed{x=\frac{-b}{2a}}$$

- If there are 2 *x*-axis intercepts, then the axis of symmetry occurs at the midpoint between the intercepts

- **The vertex** is the lowest (if $a>0$) or highest (if $a<0$) point on the graph of a quadratic function
- The vertex lies on the axis of symmetry, so we substitute the *x* value into the general formula to find the coordinate of the vertex

- **Minimum & maximum points** – the vertex is either the highest or lowest point on the graph
- If $a>0$ the parabola opens upwards $\cup$. If $a<0$ the parabola opens downwards $\cap$

TURNING POINT FORM

- **Quadratics of the form**

$$\boxed{y=a(x-h)^2+k}$$

- This is known as **turning point form**
  - *a* is dilation
  - *h* is horizontal translation
  - *k* is vertical translation
  - *(-h, k)* is the vertex
  - *x* = -*h* is the axis of symmetry

Exercise 4D Q1ajp

COMPLETING THE SQUARE

- To transform a quadratic from polynomial form into turning point form, we want to factorise as a perfect square
- When an expression is not a perfect square we must **complete the square** by "correcting" the last term
- For the quadratic in polynomial form

$$\boxed{y = ax^2 + bx + c}$$

  The corrected value of *c* must be:

$$\boxed{c = \left(\frac{b}{2}\right)^2}$$

- To sketch a quadratic by completing he square:
  - Compete the square to transform to turning point form
  - Identify the vertex
  - Identify any axis intercepts

Exercise 4E Q2aeg

POLYNOMIAL FORM

- For quadratics of the form

$$\boxed{y = ax^2 + bx + c}$$

  - c = y-intercept
- once factorised we can use the null-factor theorem to find the x-axis intercepts

$$y = a(x - m)(x - n)$$

  - m and n are x-axis intercepts
  - the constant amn is the y-axis intercept
  - the axis of symmetry is at $x = \frac{m+n}{2}$

Exercise 4F Q3ae, 4adg

IN GENERAL WHEN GRAPHING…

- **Axis intercepts** – there can be 0, 1 or 2 x-axis intercepts, and 1 y-axis intercept
- **y-axis intercept** – occurs when $x = 0$. For the general quadratic $y = ax^2 + bx + c$, the y-axis intercept is at $(0, c)$
- **x-axis intercept** – occurs when $y = 0$. Can be found by factorising, or by calculator using the graph.

- **Graphing a quadratic function** – plot the vertex and axis intercepts, then draw a smooth curve through the points

THE GENERAL QUADRATIC FORMULA

- To solve a quadratic from polynomial form, you cannot always factorise easily
- You can solve for *x* by using the **general quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Note:
  - $b^2 - 4ac > 0$ two solutions
  - $b^2 - 4ac = 0$ one solution
  - $b^2 - 4ac < 0$ no real solutions

Exercise 4G Q1ace, 2ac, 3adgj

THE DISCRIMINANT

- The discriminant is denoted by the symbol Δ

$$\boxed{\Delta = b^2 - 4ac}$$

- As noted previously:
  - $b^2 - 4ac > 0$ two solutions
  - $b^2 - 4ac = 0$ one solution
  - $b^2 - 4ac < 0$ no real solutions
- We can also use the value of Δ to identify the type of solution(s) of a quadratic equation:
  - If $\Delta = b^2 - 4ac$ is a perfect square – two **rational** solutions
  - If $\Delta = b^2 - 4ac$ $= 0$ – one **rational** solution
  - If $\Delta = b^2 - 4ac$ is **not** a perfect square – two **irrational** solutions

Exercise 4I

SOLVING QUADRATIC INEQUATIONS

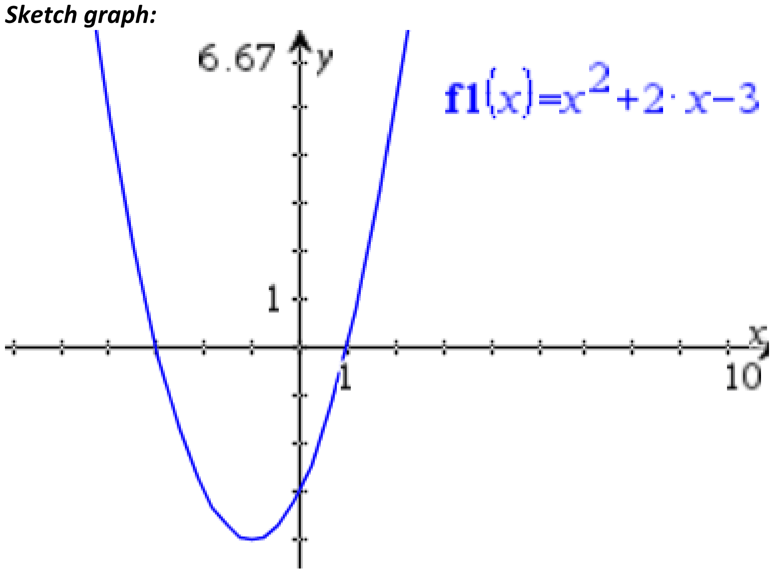
- When solving quadratic inequalities we need to:
  - Factorise and solve
  - Check the graph to determine the values for *x* which satisfy the inequality
  - State the solution(s)

For example: solve $x^2 + 2x - 3 > 0$

Factorise and solve:

$$x^2 + 2x - 3 = 0$$
$$(x-1)(x+3) = 0$$
$$\therefore x = 1 \text{ or } x = -3$$

Sketch graph:



Identify solutions:

The portion of the graph which is greater than $x = 0$ is when $x < -3$ and $x > 1$

- We can also apply our knowledge of the type of solutions based on the value of the discriminant

For example: find the values of *a* for which the equation $ax^2 + 6x + 1 = 0$

a) Has no solutions

$$\Delta < 0$$
$$\therefore 6 - 4(a)(1) < 0$$
$$6 - 4a < 0$$
$$-4a < -6$$
$$a > \frac{3}{2}$$

b) Has one solution

$$\Delta = 0$$
$$\therefore 6 - 4(a)(1) = 0$$
$$6 - 4a = 0$$
$$-4a = -6$$
$$a = \frac{3}{2}$$

Show that the equation $5x^2 + (m+6)x + m = 0$ has two irrational solutions

Δ is not a perfect square

$$(m+6)^2 - 4(5)(m) = m^2 + 12m + 36 - 20m$$
$$= m^2 - 8m + 36$$

Not a perfect square $\therefore$ two irrational solutions

Exercise 4J

SOLVING SIMULTANEOUS LINEAR AND QUADRATIC EQUATIONS

- The solution to simultaneous linear and quadratic equations occur at the points of intersection
- To solve algebraically, have both equations written with *y* as the subject, then use the method of substitution

For example: find the points of intersection of the line $y = x + 1$ and the parabola $y = x^2 + 2x - 5$

Let $x^2 + 2x - 5 = x + 1$ and solve for x

$$x^2 + 2x - 5 = x + 1$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$\therefore x = -3 \text{ or } x = 2$$

Find the coordinates by substituting the values of x into one of the original equations

$$y = (-3) + 1$$
$$y = -2$$

$$y = (2) + 1$$
$$y = 3$$

$$\therefore (-3, -2) \text{ and } (2,3)$$

- When we are given a linear equation with an unknown value, we need to apply our knowledge of the discriminant and the number of solutions in order to solve
- When the linear line is a tangent to the parabola, this means there is one intersection point and the discriminant must equal zero

For example: find the value of *c* for which $y = 2x + c$ is a tangent to the parabola $y = x^2 - 4x + 4$

Let $x^2 - 4x + 4 = 2x + c$ and collect all terms on LHS

$$x^2 - 4x + 4 = 2x + c$$
$$x^2 - 6x + 4 - c = 0$$

Let $\Delta = 0$ and solve for c

$$(-6)^2 - 4(1)(4 - c) = 0$$
$$36 - 16 + 4c = 0$$
$$20 + 4c = 0$$
$$4c = -20$$
$$c = -5$$

$\therefore$the equation of the tangent is $y = 2x - 5$

Exercise 4K

DETERMINING QUADRATIC RULES

- If you are only given a set of points – DRAW A SKETCH!
- Use any points given to set up an equation to solve for the unknown part of the equation (simultaneous equations may be required)
- Use any information from the graph to determine key points
  - Turning point/axis of symmetry – this will give you information relating to translation
  - Direction of the parabola – positive or negative dilation
  - Axis intercepts – both *y* and *x* axis intercepts are extremely useful!

For example: Find the rules of the following parabolas:

a) A parabola of the form $y = ax^2$ passing through the point (1, 5)

Substitute in the point (1, 5) and solve for *a*

$$y = ax^2$$
$$5 = a(1)^2$$
$$5 = a$$
$$\therefore y = 5x^2$$

b) A parabola of the form $y = ax^2 + c$ passing through the points (0, -1) and (3, 1)

(0, -1) is the *y* intercept

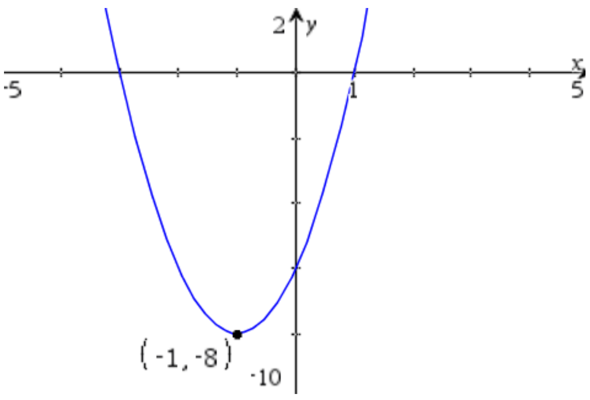
$$\therefore y = ax^2 - 1$$

Substitute in the point (3, 1) and solve for *a*

$$y = ax^2 - 1$$
$$1 = a(3)^2 - 1$$
$$1 = 9a - 1$$
$$2 = 9a$$
$$a = \frac{2}{9}$$
$$\therefore y = \frac{2x^2}{9} - 1$$

or $y = \frac{2x^2-9}{9}$

c)



Use the quadratic rule in turning point form -
$\boldsymbol{y = a(x - h)^2 + k}$**, with the turning point at (-1, -8)**

$$y = a(x+1)^2 - 8$$

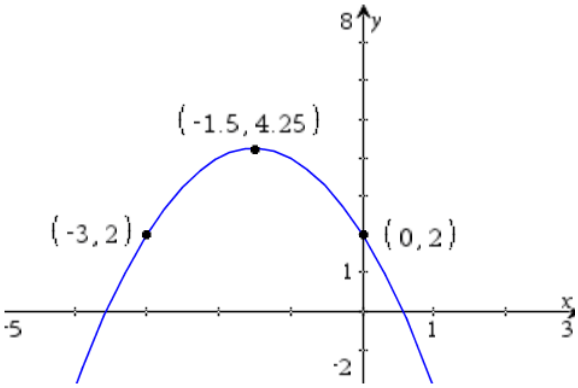
Substitute in the point (0, -6) (or any other intercept you can identify)

$$-6 = a(0+1)^2 - 8$$
$$-6 = a - 8$$
$$a = 2$$

Expand and simplify

$$y = 2(x+1)^2 - 8$$
$$y = 2(x^2 + 2x + 1) - 8$$
$$y = 2x^2 + 4x - 6$$

d)



Use the quadratic rule in polynomial form and set up a system of simultaneous equations

For (0, 2)

$$c = 2$$

For (-3, 2)

$$2 = a(-3)^2 + b(-3) + 2$$
$$0 = 9a - 3b$$

For $\left(\frac{-3}{2}, \frac{17}{4}\right)$

$$\frac{17}{4} = a\left(\frac{3}{2}\right)^2 + b\left(\frac{-3}{2}\right) + 2$$
$$\frac{9}{4} = \frac{9a}{4} - \frac{3b}{2}$$
$$9 = 9a - 6b$$

Solve for *a* and *b* by elimination

$$9 = 9a - 6b$$
$$\underline{-0 = 9a - 3b}$$
$$9 = -3b$$

$$b = -3$$

$$0 = 9a - 3(-3)$$
$$-9 = 9a$$
$$a = -1$$

$$\therefore y = -x^2 - 3x + 2$$

Exercise 4L

QUADRATIC MODELS

- We can apply our knowledge of quadratic rules and graphs to model "real life" problems and hence solve them
- Each time, take note of the information given, construct relevant equations and solve to find the solution relevant to the context of the problem.

Exercise 4M