**UNIVARIATE DATA**

1. Types of data

- **Univariate data** - data containing one variable.
- **Bivariate data** - data that more than two variables.
- **Multivariate data** - data that more than two variables.
- **Numerical data** - involves quantities that can be counted or measured.
- **Categorical data** - data that is divided into categories or groups.
- **Continuous data** - data that can have any value between two values.
- **Discrete data** - data that can take only certain fixed values.

2. Tabular Displays

- **Stem (and leaf) plots**
  - Useful for data with up to 50 observations
  - Stem can be broken into halves or fifths

- **Frequency tables**
  - Divide continuous numerical data into intervals
  - Frequency tables are an important first step in drawing a graphical display

3. Graphical Displays

- **Frequency histograms**
  - Good for continuous numerical data
  - Good for large data sets (above 50 values)
  - Vertical axis displays frequency
  - Horizontal axis shows class intervals

- **Dot plots**
  - Not good for continuous data

- **Bar charts**
  - Have gaps between the bars to indicate that the data is

- **Segmented bar charts**
  - First present the data as fractions (or percentages) of the whole data set
  - Then draw a bar (or a circle for a pie chart) and divide it into sections which are in proportion to those fractions

4. Summary Statistics

- **Median**
  - The median is a measure of the centre of the data. It is the middle value and is found at the following position in the data...
  
  \[
  \text{the } \left( \frac{n+1}{2} \right) \text{th position}
  \]

  - For example if the above formula equals 5.5 then the median is half way between the 5th and 6th values (so add the two values and divide by 2).

- **Range**
  - The range is the difference between the highest and lowest values in the data.

  \[
  \text{Range} = \text{Max} - \text{Min}
  \]

- **Quartiles** $Q_1, Q_2, Q_3$
  - The quartiles divide the data into four. The lower quartile $Q_1$ is the middle of the lower half of the data, the middle quartile $Q_2$ is the centre of the data (the median) and the upper quartile $Q_3$ is the middle of the top half of the data.

  - Tips for finding the quartiles...
  - The data must be in order
  - If there is an odd number of values in the data

- **Interquartile range (IQR)**
  - The interquartile range tells us the range of the middle 50% of values.

  \[
  \text{IQR} = Q_3 - Q_1
  \]

- **Outliers**
  - An outlier is a value that is greater than...

  \[
  Q_3 + 1.5 \times \text{IQR}
  \]

  - and is less than

  \[
  Q_1 - 1.5 \times \text{IQR}
  \]

- **Box plots**
  - A boxplot is a diagram that displays the 5-figure summary statistics of the Minimum, $Q_1$, the Median, $Q_2$, and the Maximum as shown below.

- **Mode**
  - The mode is a measure of the centre of the data, it is the most commonly occurring value in a data set.

- **Mean**
  - The mean is a measure of the centre of the data. It is less reliable than the median because it is effected if the data is skewed or contains outliers.

  - The formula to find the mean is...

  \[
  \bar{x} = \frac{\sum x}{n}
  \]

  - where $\sum x$ is the sum of all of the values in the data
  - $n$ is the number of values in the data
When using grouped data such as in a frequency table the formula is...

\[ m \] is the midpoint of the class interval
\[ f \] is the frequency

**Standard deviation**

The standard deviation is a measure of the spread of data from the mean. The symbol for standard deviation is \( s \). The larger the standard deviation, the more spread are the data from the mean.

\[
s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}
\]

where \( s \) represents sample standard deviation
\( \Sigma \) represents 'the sum of'
\( x \) represents an observation
\( \bar{x} \) represents the mean
\( n \) represents the number of observations

The square of the standard deviation (\( s^2 \)) is called the variance

**The 68–95–99.7% rule and z-scores**

The 68–95–99.7% rule for a bell-shaped curve states that:

- approx. 68% of data lie within 1 standard deviation of the mean
- approx. 95% of data lie within 2 standard deviations of the mean
- approx. 99.7% of data lie within 3 standard deviations of the mean.

This enables us to draw the following very useful diagram...

The z-score is used to measure the position of a score in a data set relative to the mean.

The formula used to calculate the z-score is...

\[
\frac{x - \bar{x}}{s}
\]

where \( x \) is the score
\( \bar{x} \) is the mean
\( s \) is the standard deviation

Scores from different data can be compared by their z-scores as it shows how extreme a value is compared to the mean and standard deviation for that data.

**Using the Calculator**

Follow these steps to generate a simple random sample.

**BIVARIATE DATA**

1. **Dependent & Independent variables**
   
   The value of the dependent variable depends on the value of the independent variable.

   On a graph the independent variable is shown on the \( x \) axis and the dependent variable is shown on the \( y \) axis.

2. **Displaying Bivariate Data**

   **Parallel boxplots** are useful for bivariate data with one numerical variable and one categorical variable with exactly two categories.

   **Two way frequency tables** are useful for bivariate data with two categorical variables. The same data can also be shown on segmented bar charts.

   **Back-to-back stem plots** are useful for bivariate data with one numerical variable and one categorical variable with more than two categories.

   **Using the Calculator**

   Use these steps to find univariate summary statistics on the calculator...

   **Populations and simple random samples**

   A population, in statistics, is a group of people (or objects) to whom you can apply any conclusions or generalisations that you reach in your investigation.

   A sample, in statistics, is...
Scatter plots are useful for bivariate data with two variables. Scatter plots display 4 aspects of the relationship between the variables...
+ Direction (positive/negative)
+ Form (linear/non-linear)
+ Strength (strong, moderate, weak)
+ Presence of outliers

3. Measuring the Correlation
Pearson's product-moment correlation coefficient \( r \) is used to measure the strength of a linear correlation between two variables. It varies from -1 to 1 as shown below.

<table>
<thead>
<tr>
<th>Strength</th>
<th>( r ) Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>0.75 to 1</td>
</tr>
<tr>
<td>Moderate</td>
<td>0.5 to 0.75</td>
</tr>
<tr>
<td>Weak</td>
<td>0.25 to 0.5</td>
</tr>
<tr>
<td>No linear</td>
<td>0</td>
</tr>
</tbody>
</table>

The coefficient of determination \( r^2 \) tells us the percentage of the variation in the dependent variable that can be explained by the variation in the independent variable.

Using the Calculator
To find \( r \) and \( r^2 \) on the calculator follow these steps...

Regression
Regression analysis involves methods of fitting lines to distributions of bivariate data.

1. Regression Methods

Fitting by Eye
When fitting by eye ensure that there is an equal number of points above and below the line.

The Three Median Method
The three median method produces a regression line that is not affected by outliers.

Divide the data into 3 equal (or at least symmetrical) groups using vertical lines then find the median points in each group...

\((x_L, y_L), (x_M, y_M), (x_U, y_U)\)

From here there are two ways to proceed...

1. Graphical Approach - Find a line from the lower median point to the upper median point then move it \( \frac{1}{3} \) of the way towards the middle median point.
2. Arithmetic Approach - find \( y = mx + c \) where

\[
m = \frac{y_U - y_L}{x_U - x_L} \\
c = \frac{1}{3}[y_L + y_M + y_U] - \frac{m}{3}[x_L + x_M + x_U]
\]

Least Squares Regression
This method of regression involves minimising the residuals which are the vertical distances from the data points to the line. It should only be used for data with a linear relationship and no outliers.

Using the Calculator
To find the three median regression line follow these steps...

LinReg(ax+b) L1, L2, Y1

2. Interpolation and extrapolation
Interpolation involves using the regression line to predict new values between existing data points.

Extrapolation involves...

In practice this involves substituting a new \( x \)-value into the formula for the line to calculate the corresponding \( y \)-value.

3. Residual Analysis
The residual is the error in the regression line given by the distance between the line and the actual data.

\[
\text{Residual} = \text{Y} - \text{Y}_{\text{predicted}}
\]

A residual plot graphs the residuals against the \( x \) values. On a residual plot any pattern at all suggests that there is some relationship that has not been fully explained by the regression line. Also...
Further Mathematics - Summary Sheet
Data Analysis

(Residual Analysis continued)

No pattern suggests that the data has a linear relationship.
A U or \( \cap \) curve suggests that the data has a non-linear relationship that can be made more linear by transforming the data.

Using the Calculator
Follow these steps to perform a residual analysis on the calculator...

4. Transforming to Linearity
This involves altering the x and/or y values to change the shape of the relationship. Use the following transformations to stretch the data as shown.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \log_{10}x ) or ( \log_{10}y )</th>
<th>( x^2 )</th>
<th>( y^2 ) or ( \frac{1}{y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After transforming the data see if it is more linear by checking the correlation coefficient.

2. Smoothing time series
Fitting a regression line to time series data with a long term trend can be done by transforming the data as shown earlier.
To make cyclical data fit a trend line better we use...
To make seasonal data fit a trend line better we use...

Moving average smoothing
Moving average smoothing involves replacing each data point with an average of itself and its adjacent points. Note that for this method the time values must be equally spaced. Here is an example...

<table>
<thead>
<tr>
<th>time</th>
<th>data</th>
<th>moving average</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>( \frac{(12 + 15 + 13)/3 = 12.3} )</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>( \frac{(12 + 15 + 13)/3 = 12.7} )</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

This is an example of 3 point smoothing because 3 points are used to calculate each smoothed value. Note that some data is when smoothing is performed.

To decide the number of points to smooth by...
+ The number of points should be much smaller than the number of values in the data set.
+ For seasonal data the number of points should equal the length of a season.
+ Always try to use an number of points.

Smoothing with an even number of points
When using an even number of points the smoothed values are not centred on existing data points. An extra step must be taken to centre the data.

<table>
<thead>
<tr>
<th>time</th>
<th>data</th>
<th>4 point smooth</th>
<th>centred data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>( \frac{(6 + 10 + 14 + 12)/4 = 10.5} )</td>
<td>( \frac{(6 + 10 + 14 + 12)/4 = 10.5} )</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>( \frac{(10 + 14 + 12 + 11)/4 = 11.75} )</td>
<td>( \frac{(10 + 14 + 12 + 11)/4 = 11.75} )</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that this extra step to centre the data produces extra data loss.

Median smoothing
Median smoothing involves replacing each data point with the median of a group of adjacent points. This can be done visually on a graph.
Median smoothing can be more effective than moving average smoothing if the data contains outliers.

Deseasonalising
Deseasonalising is a way of removing the seasonal variations from the data. This is done using the seasonal index which is a measure of how different the values in a season are from the values in the whole period.
To find the seasonal index...
1. Calculate the average for each period.
2. Divide each value by the average for the period.
3. The seasonal index is the average of these values for each season.
We can then convert between seasonalised and deseasonalised data using...

\[ \text{value} = \text{value} \times \text{seasonal index} \]
CONSTRUCTION & INTERPRETATION OF GRAPHS

1. Linear graphs

Linear equations

A graph that consists of a straight line can be described by this equation...

\[ y = mx + c \]

where

- \( m \) is the gradient
- \( c \) is the y-intercept

The gradient \( m \) can be found using the following...

\[ m = \frac{\text{Rise}}{\text{Run}} \]

or \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

where \((x_1, y_1)\) and \((x_2, y_2)\) are two points on the line

Sketching linear graphs

To sketch a linear graph plot any two points on the line then draw a line that joins them. The three main methods of sketching a linear graph are...

- **The x- and y-intercept method**: find the x-intercept (by substituting \( y = 0 \) into the equation) then find the y-intercept (by substituting \( x = 0 \) into the equation).
- **Graphing over a defined interval**: substitute any two \( x \)-values into the equation to find two points on the line.
- **The gradient-intercept method**: put the equation for the line into the form \( y = mx + c \) then draw a line through the y-intercept (\( c \)) with the gradient \( m \).

2. Line segments & step functions

A variable that can only take specific values is called a discrete variable (e.g., the number of students in a class can only be whole numbers).

A variable that can take any value in a range is called continuous.

Graphs of discrete data are often made up of separate dots, or line segments that skip some \( x \) values.

Graphs of continuous data can have one continuous line or they can include line segments that either join up or have jumps or steps.

3. Simultaneous equations

If two lines intersect then there is a simultaneous solution where the same coordinates \((x, y)\) satisfy the equations of both graphs. This point can be found in a number of ways...

- **Graphically**: by reading the point from a graph
- **Algebraically**: using the substitution method which is the best method when the equations are in the form \( y = mx + c \)
- **Algebraically**: using the elimination method which is the best method when equations are in the form \( ax + by = c \)
- **Using a graphics calculator**: by plotting the graphs and then pressing \( \text{[CALC]} \) and choosing \( \text{5:intersect} \)

Simultaneous equations are used in business where one line is used to show costs and the other revenue. Here the intersection point represents the \( \text{break-even point} \) where costs and revenues are equal.

4. Non-linear graphs

Interpreting non-linear graphs

A non-linear graph is a graph where the gradient is not constant.

- The gradient or slope at a point on the line shows the rate of change of the \( y \) variable compared to the \( x \) variable
- The y-intercept is the value when \( x = 0 \) and often represents the initial or starting value

Referring to the examples above

- in A the rate of change is increasing
- in B the rate of change is decreasing
- so in C we can say that the rate of change is decreasing at first and then

Constructing non-linear relations and graphs

Simple non-linear graphs can be modelled with equations in the form...

\[ y = kx^n \]

The value of \( n \) can be determined by recognising the general shape of the graph as follows...

<table>
<thead>
<tr>
<th>( n )</th>
<th>( y )</th>
<th>( 0 )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = -2 )</td>
<td>( y )</td>
<td>( 0 )</td>
<td>( 6 )</td>
<td>( 24 )</td>
<td>( 54 )</td>
<td>( 96 )</td>
<td>( 150 )</td>
</tr>
</tbody>
</table>

To draw a graph from a non-linear equation

- Construct a table of values and calculate the \( y \) value for a range of \( x \) values such as the one shown below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 0 )</td>
<td>( 6 )</td>
<td>( 24 )</td>
<td>( 54 )</td>
<td>( 96 )</td>
<td>( 150 )</td>
</tr>
</tbody>
</table>

- Plot these points on a graph and then draw a curve that joins them

To find the equation of a relationship

- Look at the shape of the graph to determine \( n \) if necessary
- Construct a table of values for \( x \) and \( y \) and add a row to calculate values of \( x^n \). For the table shown above this row would look like...

| \( x^n \) | \( 0 \) | \( 4 \) | \( 36 \) | \( 100 \) |

- Plot \( y \) versus \( x^n \). This should be a straight line.
- Now calculate the gradient which becomes \( k \) in the formula \( y = kx^n \)
**LINEAR INEQUATIONS & LINEAR PROGRAMMING**

1. **Graphing linear inequations**

   Linear inequations are written in the form...
   
   \[ ax + by < c \]

   where the inequality sign \(<\) can be \(<\), \(\leq\) or \(\geq\)

   The solution to a linear inequation is a region (or half-plane) either above or below the line.

   When graphing a linear inequation we shade the area that is not a solution, in other words we shade the points for which the inequation is false.

   To sketch the graph of a linear inequation follow these steps
   
   1. Plot the line for the corresponding linear equation (this is the boundary of the solution region)
   2. The line is drawn with either a dotted or a solid line...
   3. If the inequality sign is \(\leq\) or \(\geq\) then points on the line are part of the solution region and a solid line is drawn
   4. If the inequality sign is \(<\) or \(>\) then points on the line are not part of the solution region and a dotted line is drawn
   5. Pick a test point above or below the line
   6. If at this point the inequation is true then shade the other side of the line
   7. If it is false then shade the side of the line which includes the test point.

   For example the graph \(3x - 4y < 12\) looks like this..

2. **Simultaneous linear inequations**

   The simultaneous solution for a group of linear inequations is the region where all of the inequations are true.

   To find the solution region for a group of linear inequations
   
   1. graph the inequations on the same set of axes
   2. the solution region will be the region on the graph that is shaded

   The solution region will be a polygon the number of sides of which can be up to the number of inequations in the group.

3. **Linear programming problems**

   Linear programming problems are made up of three components...
   
   + a set of constraints (in Further Maths we consider problems with two variables labelled \(x\) and \(y\))
   + a set of decision variables in the form a linear inequations
   + an objective function which needs to be maximised (for example profit) or minimised (for example costs).

   To solve general linear programming problems follow these steps
   
   1. define the decision variables \(x\) and \(y\)
   2. define the constraints by creating a set of linear inequations to represent the situation
   3. graph the constraints as a set of simultaneous linear inequations
   4. determine the corners or vertices of the solution region by treating the intersecting lines as simultaneous linear equations
   5. define the objective function
   6. choose the vertex point that best achieves the objective by minimising or maximising the objective function.

   **Manufacturing Problems**

   These are simpler than the other types of problems we consider and generally involve the manufacture of two products for which resources such as money, parts and staff must be shared.

   The decision variables are usually
   
   + \(x\) is the number of one product manufactured
   + \(y\) is the number of the other product manufactured

   **Blending problems**

   These involve combining different raw materials into a single composite product.

   These appear to involve three variables ...
   
   \(x, y\) and \(z\)
   
   ... but because we are given a total
   
   \(x + y + z = 50\)
   
   ... one is expressed in terms of the other two
   
   \(z = 50 - x - y\)

   **Transportation problems**

   These involve minimising shipping costs when transporting from two locations to two destinations. To work out the constraints for these problems it is useful to create a table such as the one below.

<table>
<thead>
<tr>
<th>Source A</th>
<th>Source B</th>
<th>Total Required</th>
</tr>
</thead>
<tbody>
<tr>
<td>Destination 1</td>
<td>(x)</td>
<td>70 - (x)</td>
</tr>
<tr>
<td>Destination 2</td>
<td>(y)</td>
<td>(z)</td>
</tr>
<tr>
<td>Maximum supply available</td>
<td>100</td>
<td>150</td>
</tr>
</tbody>
</table>

   Note the following tips
   
   + the decision variable \(x\) and \(y\) represent the number shipped from one source to each destination
   + the totals are then used to find the number shipped from the other source to each destination
   + the maximum supply from each source is then used to define the constraints

   \(eg: x + y \leq 100\)
**GEOMETRY**

1. Angles, triangles & polygons

**Geometric notation and rules**

- This notation is used to show parallel lines, lines of equal length and equal angles.

- Angles can be called acute, obtuse or reflex depending on their size.

- A right angle is a 90 degree angle; a straight angle is a straight line (an angle of 180 degrees).

- Also be aware of the following rules of geometry:
  - For a regular polygon (all sides and angles equal) with n sides...
    - the exterior angles are given by \( \frac{360^\circ}{n} \)
    - the interior angles are therefore \( \frac{180n - 360}{n} \).

2. Area and perimeter

- Perimeter is the distance around a closed figure.
- Circumference is the perimeter of a circle.

- The following diagrams illustrate unit conversion for...
  - Length
    - mm → cm → m → km
    - \( \times 10 \) \( \times 100 \) \( \times 1000 \)
  - Area
    - \( mm^2 \) \( cm^2 \) \( m^2 \) \( km^2 \)
    - \( \times 10^2 \) \( \times 100^2 \) \( \times 1000^2 \)

**Total Surface Area**

- Total surface area (TSA) of some common objects are as follows:

- Volume

- Volume is the amount of space occupied by a 3-dimensional object.
- The units of volume (mm\(^3\), cm\(^3\), mL, L & m\(^3\)) are converted as follows:
  - \( 1000 \) mm\(^3\) = 1 cm\(^3\)
  - \( 1000000 \) mm\(^3\) = 1 m\(^3\)
  - 1 cm\(^3\) = 1 millilitre (mL)
  - 1 litre = 1000 mL
  - 1000 litres = 1 m\(^3\)

- Volume of a prism, \( V_{\text{prism}} = \text{area of uniform cross-section} \times \text{height} \)

- Volume of a pyramid = \( \times \text{area of cross-section at the base} \times \text{height} \)

- Volume of a sphere is \( V_{\text{sphere}} = \frac{4}{3} \pi r^3 \)

- Volume of a composite object = sum of the volumes of the individual common prisms, pyramids or spheres.

\[ V_{\text{composite}} = V_1 + V_2 + V_3 + \ldots \]

or \( V_{\text{composite}} = V_1 - V_2 \ldots \)
4. Similar figures

- Two objects that have the same shape but different size are said to be similar.
- For two figures to be similar, they must have the following properties:
  - The ratios of the corresponding sides must be equal (SSS).
  - The corresponding angles are equal (AAA).

Scale factor (k)

- Linear scale factor: \( k = \frac{\text{length of image}}{\text{length of original}} \)
- For enlargements, \( k \) is above 1, for reductions \( k \) is between 0 & 1.
- For \( k = 1 \), the figures are exactly the same shape and size and are referred to as congruent.

Similar triangles

- Two triangles can be said to be similar using the SSS and AAA rules from above and also if two corresponding pairs of sides are in the same ratio and the included angles are equal (SAS).

Area and volume scale factors

- Area scale factor = \( \frac{\text{area of image}}{\text{area of original}} \)
- Volume scale ratio or factor = \( \frac{\text{volume of image}}{\text{volume of original}} \)
- Asf = square of linear scale factor (lsf) = \( k^2 \)
- Vsf = cube of linear scale factor (lsf) = \( k^3 \)

To find an unknown value using information from two similar figures:

- Clearly identify the known corresponding measurements (length, area or volume) of the similar shapes.
- Establish a scale factor (linear, area or volume) using known pairs of measurements.
- Convert to an appropriate scale factor to determine the unknown measurement.
- Use the scale factor and ratio to evaluate the unknown.

TRIGONOMETRY

1. Right-angled triangles

Pythagoras' theorem

\[ c^2 = a^2 + b^2 \]

Three-dimensional Pythagoras' theorem

- To solve problems involving three-dimensional Pythagoras' theorem:
  1. Draw and label an appropriate diagram.
  2. Identify the right angles.
  3. Identify right-angled triangles that enable the information given to be used to find the unknown value(s).

Trigonometric ratios

- Remember SOHCAHTOA

\[
\begin{align*}
\sin \theta &= \frac{\text{Opposite}}{\text{Hypotenuse}} \\
\cos \theta &= \frac{\text{Adjacent}}{\text{Hypotenuse}} \\
\tan \theta &= \frac{\text{Opposite}}{\text{Adjacent}}
\end{align*}
\]

2. Non-right-angled triangles

- An angle found with the sine rule is ambiguous when the smaller known side is opposite the known angle.
- To find the obtuse angle that returns the same sine ratio for a given acute angle use...
  \[ \text{obtuse angle} = 180^\circ - \text{acute angle} \]

The sine rule:

- The sine rule is used when:
  1. two angles and one side are given
  2. two sides and a non-included angle are given.
- If two angles are given, simply calculate the third angle, if needed, using:
  \[ C = 180^\circ - (A + B) \]

Ambiguous case of the sine rule

- The cosine rule

- The cosine rule is used...
  1. to find an angle when all three sides are given
  2. to find a side when two sides & the included angle are given.

Area of triangles

- If the height and base are known use...
  \[ \text{Area}_{\text{triangle}} = \frac{1}{2} \times \text{Base} \times \text{Height} \]
- If two sides and the included angle are known use...
  \[ \text{Area}_{\text{triangle}} = \frac{1}{2}ab \sin C \]
- If all three sides are known use Heron's formula:
  \[ \text{Area}_{\text{triangle}} = \sqrt{s(s-a)(s-b)(s-c)} \]
  where \( s \) is the semi-perimeter \( s = \frac{a+b+c}{2} \)
1. Degrees and Minutes

- Angles and bearings are measured in degrees and minutes.
- 60 minutes = 1 degree
- To convert decimal degrees to minutes multiply by 60
  \[ 35.4° = 35° + (0.4 \times 60′) = 35° 24′ \]
- To convert minutes to decimal degrees divide by 60
  \[ 35° 24′ = 35° + (24′ ÷ 60′) = 35.4° \]

2. Angles of elevation and depression

- These are vertical angles
- An angle of depression is an angle below horizontal
- An angle of elevation is an angle above

3. Bearings

- These are horizontal angles

   **Standard compass bearings**
   - These use letters such as N, NW and SSE
   - Example:
     - N20°W
     - S80°E

   **Non-standard compass bearings**
   - These start at north or south, then turn through an angle towards east or west, for example N20°W, S80°E.

   **True bearings:**
   - These start at north and then turn through an angle in a clockwise direction, for example 157°T, 030°T, 287°T.

4. Navigation & specification of locations

- When solving navigation problems, in most cases the angle laws will need to be used.
- When determining a bearing, be clear on where the direction is taken from and to (the starting and finishing points).
- There is a 180° difference between the bearing of A from B compared to the bearing of the return, that is, of B from A.

5. Triangulation

- Triangulation involves finding dimensions in inaccessible regions.
- Sine and cosine rules may be used if:
  (a) the distance between two locations is known and
  (b) the direction from the two locations to a third is known.
- Alternatively, we may use similarity when two similar triangles are given.

6. Contour maps

- A contour map represents the shape of the terrain.
- Contour lines join locations that are at the same height (or altitude) above sea level or a reference point.
  - Contour lines that are close together indicate steep terrain.
  - Contour lines that are far apart indicate gentle slopes.

   **Finding the average slope**
   - The distance (rise) between two locations can be found from the difference in the values of the two contour lines.
   - Distances (run) are found by measuring distances on the map and converting them to actual distances using the map scale.
   - The slope is then given by the formula...
     \[ m = \frac{\text{Rise}}{\text{Run}} \]
   - To find the angle of the slope use...
     \[ \tan \theta = \frac{O}{A} = \frac{\text{Rise}}{\text{Run}} \]
**LOANS & INVESTMENTS**

Growth is when a quantity increases over time. 
Decay is when a quantity decreases over time.

Both growth and decay can be modelled in 3 ways:
1. Straight line or simple interest.
2. Exponential (curved line) or compound interest.
3. Models involving payments over time.

1. **Simple interest**

Simple interest is calculated as a percentage of the amount borrowed. The amount borrowed is called the principal.

\[ A = P + I \]

where
- \( A \) is the final amount
- \( P \) is the principal
- \( I \) is the interest

\[ I = P r T \]

where
- \( r \) is the interest rate (per period)
- \( T \) is the time (number of periods)

Important: make sure you use the same unit of time for \( T \) and \( r \).

2. **Compound interest**

Is when interest is earned on any interest that has already been paid. As a result the amount of interest earned each period increases over time.

\[ A = P R^n \]

where
- \( A \) is the final amount
- \( P \) is the principal
- \( n \) is the number of compounding periods
- \( R \) is the compounding factor

\[ R = 1 + \frac{r}{100} \]

where
- \( r \) is the interest rate (per compounding period)

Using the above formulas we can find the total amount. To find the interest we then use...

\[ I = A - P \]

Important: we can use these formulas to find \( A, P \) or \( R \). Finding \( n \) can be difficult, so it is best to use the calculator (see below).

**Using the Calculator**

How to solve compound interest problems with the TVM solver...

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3. **Reducing balance loans**

**Annuities** are regular payments. Compound interest loans that involve payments are called annuities.

Some definitions...
- The **balance** is the amount still owing (at some time)
- The **term** is the time from when the loan is taken out to when it is paid off.
- To **discharge** a loan is to pay it off (often before the end of the term)

These can be investigated using the Annuities Formula:

\[ A_n = P R^n - Q (R^n - 1) \]

where
- \( A_n \) is the amount owing after \( n \) periods
- \( P \) is the principal
- \( R \) is the compounding factor
- \( Q \) is the payment amount per period
- \( n \) is the number of periods

**Using the Calculator**

Reducing balance loans can be solved using the TVM solver. Use the same method used for compound interest problems but include the payment value (\( Q \)) next to PMT. Note that for reducing balance loans PMT should have the same sign as \( Q \).

4. **Hire purchase**

In hire-purchase arrangements a deposit is often paid, before flat-rate (simple interest) is paid over a fixed period.

To find the amount to be paid in each installment...
1. Deduct the deposit
2. Calculate and add the interest using the simple interest formula
3. Divide by the number of installments

The total cost can be found using either of the following formulas...

\[ \text{Price (including deposit)} + \text{Interest} \]

or

\[ \text{Deposit} + (\text{installment} \times \text{number of installments}) \]

5. **Effective rate of interest**

The effective rate of interest is the equivalent reducing balance rate for a flat rate loan.

\[ \text{Effective rate} = \left(1 + \frac{r}{100}\right)^n - 1 \]

where \( n \) is the number of payments

6. **Perpetuities**

Perpetuities are investments which provide regular payments that come from the interest earned but do not effect the principal. This means that the payments can theoretically continue forever.

The perpetuity formula:

\[ Q = \frac{P r}{100} \]

**Using the Calculator**

To find Perpetuities with the TVM Solver set FV to equal the principal and set PMT to the same value but with a negative sign.
Further Mathematics - Summary Sheet
Business Maths

7. Annuity investments

Annuity investments are investments where the principal grows as a result of regular payments as well as interest earned. An example of an annuity investment is superannuation.

To solve annuity investment problems we alter the annuities formula so that the payments are added to the principle (instead of being deducted from an amount owed).

\[ A_n = P \frac{R^n - 1}{R - 1} + Q \]

**Using the Calculator**

To solve annuity investment problems with the TVM Solver give the payment amount (PMT) the same sign as the principal (PV).

---

**FINANCIAL TRANSACTIONS AND ASSET VALUE**

1. Bank accounts

Normal savings accounts calculate interest based on either...
- The minimum monthly balance or...
- The daily balance

The interest is added at a specified time (often monthly).

Credit cards can have either...
- An annual fee and an interest free period (up to 55 days) or...
- No interest free period but no annual fee

When an amount is owed on a credit card, the bank requires a minimum payment each month. This is calculated according to some system such as...
- For a balance of less than $25, the payment is the balance
- For a balance above $25, the payment is of $25 or 1.5% of the balance, whichever is greater.

Note: If the closing balance is greater than the limit on the card, then the minimum payment must also include this excess.

The following formula can be used to find the balance on a credit card:

\[ \text{Closing Balance} = \text{Opening Balance} - \text{Payments} + \text{Purchases} + \text{Interest} \]

After the interest free period, a card will revert to no interest free period for all purchases.

2. Financial computations

Discounts

\[ \text{Discount} = \text{Original Price} \times \% \text{ discount} \]

Goods & Services Tax (GST)

For problems involving sale or purchase price use either of...
- Sale price = Purchase price × (10% of Purchase price)
- Sale price = Purchase price × 

For problems involving the GST amount choose from...
- GST amount = Sale price – Purchase price
- GST amount = \[ \frac{\text{Sale price}}{11} \]

3. Depreciation

- The estimated value of an item at a point in time is called the **book value**.
- The **scrap value** is the remaining value of an item at the end of its useful life
- The **effective life** is the time taken for the book value to reach the scrap value

**Flat rate depreciation**

This is also called prime cost depreciation and is based on simple interest.

\[ \text{Flat rate depreciation} = P \times r \times T \]

where
- \( P \) is the cost price
- \( r \) is the rate of depreciation per unit of time (a percentage)

**Reducing balance depreciation**

This is based on the compound interest model.

\[ \text{Reducing balance depreciation} = PV \times (1 - r)^T \]

where
- \( PV \) is the present value
- \( r \) is the rate of depreciation per unit of time (a percentage)

**Using the Calculator**

To use the TVM solver for reducing balance depreciation problems set PMT to zero, set R and PV as negative and FV as positive.

**Unit cost depreciation**

Here depreciation is calculated based on the estimated maximum output of an item. For example, the number of kilometers a car can drive in its useful life.

\[ \text{Amount of depreciation} = \text{Amount of use} \times \text{depreciation rate} (\$ \text{ per use}) \]

4. Inflation

Inflation is a measure of the average increase in prices from one year to the next. It works like compound interest.

\[ A = P \left(1 + \frac{r}{100} \right)^T \]

where
- \( A \) is the price after time \( T \)
- \( P \) is the original price
- \( T \) is the time in years
- \( r \) is the inflation rate
1. Basic concepts of networks

A network is a collection of objects that are connected to each other in some specific way. In mathematics each object is called a node or vertex and the lines connecting them are edges or paths.

The degree of a vertex is the number of edges connecting to it. A vertex with degree 0 has no edges and is called an isolated vertex. A loop connects a vertex to itself and adds to its degree.

A Network can be described three ways...

1. Graphically:

   ![Graph Example]

2. By listing the vertices then listing the edges according to the vertices they connect to:

   \[ V = \{ A, B, C, D, E \} \]

   \[ E = \{ (A,B), (A,C), (A,D), (B, C), (B, D), (B, D), (C, E), (D, E), (E, E) \} \]

   Note: - An isolated vertex is not in the list of edges
   - The number of pairs in E is the number of edges
   - The number of times a vertex is listed in E is its degree

3. Using matrices:

   \[
   \begin{bmatrix}
   A & B & C & D & E \\
   0 & 1 & 1 & 0 & 0 \\
   B & 0 & 1 & 0 & 2 & 0 \\
   C & 1 & 0 & 0 & 1 & 0 \\
   D & 1 & 2 & 0 & 0 & 1 \\
   E & 0 & 0 & 1 & 1 & 2 \\
   \end{bmatrix}
   \]

   Note: - The sum of a row or column gives the degree of that vertex
   - A row or column containing only 0s indicates an isolated vertex
   - These matrices are diagonally symmetrical

2. Planar graphs and Euler’s formula

- A planar graph has no edges that cross.
- A degenerate graph has no edges.
- A complete graph has all vertices directly connected to all other vertices.

Planar graphs can be used to represent 2 and 3 dimensional objects:

- This graph represents a cube.
- The regions represent the faces of the cube and are labelled using Roman numerals.
- The infinite region (VI) represents the side of the cube that cannot be seen.
- The degree of a face is the number of edges defining it.

Euler’s formula states that for planar graphs...

\[ V - E + F = 2 \]

where

- V is the number of vertices
- E is the number of edges
- F is the number of faces

3. Paths and circuits

A connected graph has a path between all possible pairs of vertices.

- A path uses every edge exactly once. For a path to exist...
  - All vertices must be of even degree or...
  - There must be exactly two vertices of odd degree and the path must start on and finish on

- An Euler circuit uses every edge exactly once and starts and finishes on the same vertex. For an Euler circuit to exist...
  - All vertices must be of even degree

To find an Euler circuit...

1. Choose a starting vertex
2. Find the smallest circuit starting from and returning to this vertex
3. For each vertex in this circuit find the smallest circuit starting from and returning to that vertex.
4. Continue adding these small circuits until all edges have been covered.
5. Describe the circuit by joining the subcircuits at their intersection points.

- A Hamiltonian path uses every vertex exactly once. For a Hamiltonian path to exist...
  - There can be up to two vertices of degree

- A Hamiltonian circuit uses every vertex exactly once and starts and ends on the same vertex. For a Hamiltonian circuit to exist...
  - There must be no vertices of degree

4. Trees and their applications

- A graph has at least two vertices and one edge.
- A subgraph contains some of the edges and vertices from a graph.
- A minimum subgraph is a subgraph containing only 2 vertices and one edge.
  - In a subgraph the edges are assigned a quantity such as distance, time or cost.
- A cycle is a subgraph containing a path which starts and finishes at the same vertex.
- A tree is a connected subgraph which does not contain any loops, parallel edges or cycles.

The shortest path between two vertices will always be a tree.

To find the shortest path between two vertices...

1. From a starting vertex list the shortest path to all of the vertices directly connected to it.
2. Choose the next closest vertex to those already listed and find the shortest path to it.
3. Repeat this until the destination vertex has been reached.

- A tree is a tree that includes all of the vertices in a graph
- A minimum tree is the one with the smallest value for the sum of the included edges.

To find a minimum spanning tree (Prim’s algorithm)...

1. Choose the edge with the smallest value.
2. Choose the smallest edge connecting to the first one.
3. Continue adding the smallest edge connected to those already chosen until all vertices in the graph are included in the tree.

This can also be used to find a maximum spanning tree just by choosing the largest edge each time.
1. Directed graphs

An activity chart lists the different tasks that need to be completed in an operation:

<table>
<thead>
<tr>
<th>label</th>
<th>activity</th>
<th>predecessor</th>
<th>time(min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>eat breakfast</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>download email</td>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>read email</td>
<td>B</td>
<td>2</td>
</tr>
</tbody>
</table>

This can also be represented as a network:

The arrows on the edges indicate that this is a directed graph. In a directed graph the paths must be taken in a certain direction.

To draw a directed graph:
1. List any activities that have no predecessors first.
2. Add activities to the graph after their predecessors have been added.

Don’t forget: the activities are the edges not the vertices.

2. Critical path analysis

The critical path is the path along which none of the activities can be delayed. This can be found using critical path analysis which involves forwards and then backwards scanning through the network.

For critical path analysis to be performed the network must meet the following conditions:
1. Two nodes can be connected directly by a maximum of one edge (no parallel edges).
2. An activity can be rerepresented by exactly one edge only.

If these rules are not met by the network this can sometimes be solved by inserting a dummy edge.

By forward scanning through a network we can find the earliest start time for each activity (the time at which all prior activities have been completed). The earliest start time for the last (or finish) node is the earliest time of the whole project.

After the forward scan we can then find the latest start time for each activity by backwards scanning through the network. The latest start time is the latest time that an activity can be delayed until without delaying the completion of the project.

To complete the backwards scanning step:
1. Enter the same number in the triangle and box at the end of the project.
2. Move backwards through the network. For each path subtract the activity time from the latest finish time to find the latest start time. In other words...

   start box = ...

3. Where two or more paths come together choose the largest value as the latest start time.

   Here activity B has a float time of 3 minutes. This means that the activity could be delayed for 3 minutes without affecting the completion time of the project.

   float time = end box - start box

   After backwards scanning, the critical path can be seen by the path where

3. Network Flow

In network flow diagrams the start node is called the source & the end node is the sink. The edges should show the direction and quantity of flow.

The outflow at a node will be either the sum of the edges flowing into it or the sum of the edges flowing out of it, whichever is smaller.

The flow capacity of the network (also called the maximum flow) is the largest amount that can flow through the whole network.

One way to find the maximum flow through a network is to cut it off at some point and see how much is flowing across the cut. This must be repeated for all possible cuts, then the maximum flow will be given by the minimum cut.

Note: At each cut, only count the flow that goes from inside the cut (on the side with the source) to outside the cut (on the side with the sink).

Also note that for a cut to be valid it must satisfy

4. Assignment problems & bipartite graphs

In bipartite graphs the nodes can be separated into two categories... supply and demand.

The best combination of edges from the supply to the demand nodes is called the optimal allocation. This involves minimising some quantity such as cost or time, or maximising a quantity such as profit.

To find the optimal allocation, row reduction can be used. This involves presenting the information in a matrix and subtracting the smallest value in each row from all of the values in the row. Then draw the smallest number of lines required to cover the zeros. If the number of lines is equal to the number of demand nodes then the optimal allocation has been found.

If this method does not succeed we can use the Hungarian Algorithm:
1. Try a row reduction. If the resulting matrix does not show an optimal allocation apply a column reduction to it.
2. If there is still not an optimal allocation we use Hungarian algorithm. Start with the matrix from the end of step 1 and find the smallest uncovered number. Add this number to all of the covered numbers. If a number is at the intersection of two lines add the number twice.
3. Subtract the overall smallest number from all of the numbers in the matrix. Attempt an allocation. If it fails go to step 3 and repeat.

Note: if the problem involves maximising a value choose the largest value at each step instead of the smallest.

Reachability & dominance

Reachability describes how easy it is to get from one vertex in a directed graph to another. A one-stage pathway involves two vertices directly linked by an edge. In a two-stage pathway, two edges are used. The number of routes entering a vertex is its indegree & the number leaving is its outdegree.

The pathways can be shown in matrices. The outdegree is the sum of the numbers in a column & the indegree is the sum of a row. By adding the matrices we can find the node with the largest outdegree called the dominant vertex.