CHAPTER

3

Quadratics

Objectives

- To recognise and sketch the graphs of **quadratic relations**.
- To determine the **maximum** or **minimum** values of a quadratic relation.
- To **solve quadratic equations** by factorising, completing the square and using the general formula.
- To apply the **discriminant** to determine the nature and number of roots of quadratic relations.
- To apply quadratic relations to solving problems.

A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \qquad (n \in N)$$

where $a_0, a_1 \ldots a_n$ are numbers called **coefficients**.

The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient.

Examples

i $y = 2x + 3$ is a polynomial of degree 1.
ii $y = 2x^2 + 3x - 2$ is a polynomial of degree 2.
iii $y = -x^3 + 3x^2 + 9x - 7$ is a polynomial of degree 3.

First degree polynomials, otherwise called **linear** relations, have been discussed in Chapter 2.

In this chapter second degree polynomials will be investigated. These are called **quadratics**.

3.1 Expanding and collecting like terms

Finding the x-axis intercepts, if they exist, is a requirement for sketching graphs of quadratics. To do this requires the solution of quadratic equations, and as an introduction to the methods of solving quadratic equations the basic algebraic processes of expansion and factorisation will be reviewed.

Example 1

Simplify $2(x-5)-3(x+5)$, by first expanding.

Solution

$$\begin{aligned} 2(x-5)-3(x+5) &= 2x-10-3x-15 && \text{Expand each bracket.} \\ &= 2x-3x-10-15 && \text{Collect like terms.} \\ &= -x-25 \end{aligned}$$

Example 2

Expand $2x(3x-2)+3x(x-2)$.

Solution

$$\begin{aligned} 2x(3x-2)+3x(x-2) &= 6x^2-4x+3x^2-6x \\ &= 9x^2-10x \end{aligned}$$

For expansions of the type $(a+b)(c+d)$ proceed as follows:

$$\begin{aligned} (a+b)(c+d) &= a(c+d)+b(c+d) \\ &= ac+ad+bc+bd \end{aligned}$$

Example 3

Expand the following:

a $(x+3)(2x-3)$ **b** $(x-3)(2x-2\sqrt{2})$

Solution

a $(x+3)(2x-3)$

$$\begin{aligned} &= x(2x-3)+3(2x-3) \\ &= 2x^2-3x+6x-9 \\ &= 2x^2+3x-9 \end{aligned}$$

b $(x-3)(2x-2\sqrt{2})$

$$\begin{aligned} &= x(2x-2\sqrt{2})-3(2x-2\sqrt{2}) \\ &= 2x^2-2\sqrt{2}x-6x+6\sqrt{2} \\ &= 2x^2-(2\sqrt{2}+6)x+6\sqrt{2} \end{aligned}$$

Example 4

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Expand $(2x-1)(3x^2+2x+4)$.

Solution

$$\begin{aligned} (2x-1)(3x^2+2x+4) &= 2x(3x^2+2x+4)-1(3x^2+2x+4) \\ &= 6x^3+4x^2+8x-3x^2-2x-4 \\ &= 6x^3+x^2+6x-4 \end{aligned}$$

Consider the expansion of a perfect square, $(x + a)^2$.

$$\begin{aligned}(x + a)^2 &= (x + a)(x + a)\\ &= x(x + a) + a(x + a)\\ &= x^2 + ax + ax + a^2\\ &= x^2 + 2ax + a^2\end{aligned}$$

Thus the general result can be stated as:

$$(x + a)^2 = x^2 + 2ax + a^2$$

Example 5

Expand $(3x - 2)^2$.

Solution

$$\begin{aligned}(3x - 2)^2 &= (3x)^2 + 2(3x)(-2) + (-2)^2\\ &= 9x^2 - 12x + 4\end{aligned}$$

Consider the expansion of $(x + a)(x - a)$.

$$\begin{aligned}(x + a)(x - a) &= x(x - a) + a(x - a)\\ &= x^2 - ax + ax - a^2\\ &= x^2 - a^2\end{aligned}$$

Thus the expansion of the difference of two squares has been obtained:

$$(x + a)(x - a) = x^2 - a^2$$

Example 6

Expand:

a $(2x - 4)(2x + 4)$ **b** $(x - 2\sqrt{7})(x + 2\sqrt{7})$

Solution

a $$\begin{aligned}(2x - 4)(2x + 4) &= (2x)^2 - (4)^2\\ &= 4x^2 - 16\end{aligned}$$

b $$\begin{aligned}(x - 2\sqrt{7})(x + 2\sqrt{7}) &= x^2 - (2\sqrt{7})^2\\ &= x^2 - 28\end{aligned}$$

Example 7

Expand $(2a - b + c)(2a - b - c)$.

Solution

$$\begin{aligned}(2a - b + c)(2a - b - c) &= ((2a - b) + c)((2a - b) - c)\\ &= (2a - b)^2 - c^2\\ &= 4a^2 - 4ab + b^2 - c^2\end{aligned}$$

Exercise 3A

1 Expand each of the following:

a $2(x - 4)$ b $-2(x - 4)$ c $3(2x - 4)$
d $-3(4 - 2x)$ e $x(x - 1)$ f $2x(x - 5)$

2 Collect like terms in each of the following:

a $2x + 4x + 1$ b $2x - 6 + x$ c $3x + 1 - 2x$ d $-x + 2x - 3 + 4x$

Example 1
3 Simplify each of the following by expanding and collecting like terms:

a $8(2x - 3) - 2(x + 4)$ b $2x(x - 4) - 3x$
c $4(2 - 3x) + 4(6 - x)$ d $4 - 3(5 - 2x)$

Example 2
4 Simplify each of the following by expanding and collecting like terms:

a $2x(x - 4) - 3x$ b $2x(x - 5) + x(x - 5)$ c $2x(-10 - 3x)$
d $3x(2 - 3x + 2x^2)$ e $3x - 2x(2 - x)$ f $3(4x - 2) - 6x$

Example 3
5 Simplify each of the following by expanding and collecting like terms:

a $(3x - 7)(2x + 4)$ b $(x - 10)(x - 12)$ c $(3x - 1)(12x + 4)$
d $(4x - 5)(2x - 3)$ e $(x - \sqrt{3})(x - 2)$ f $(2x - \sqrt{5})(x + \sqrt{5})$

Example 5
6 Simplify each of the following by expanding and collecting like terms:

a $(x - 4)^2$ b $(2x - 3)^2$ c $(6 - 2x)^2$
d $\left(x - \frac{1}{2}\right)^2$ e $(x - \sqrt{5})^2$ f $(x - 2\sqrt{3})^2$

7 Simplify each of the following by expanding and collecting like terms:

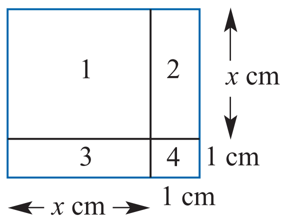
a $(2x - 3)(3x^2 + 2x - 4)$ b $(x - 1)(x^2 + x + 1)$
c $(6 - 2x - 3x^2)(4 - 2x)$ d $(x - 3)(x + 3)$
e $(2x - 4)(2x + 4)$ f $(9x - 11)(9x + 11)$
g $(5x - 3)(x + 2) - (2x - 3)(x + 3)$ h $(2x + 3)(3x - 2) - (4x + 2)(4x - 2)$
i $(x - y + z)(x - y - z)$ j $(x - y)(a - b)$

8 Find the area of each of the following by:

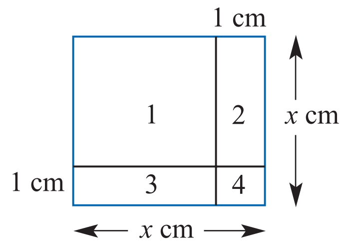
i finding the area of the four 'non overlapping' rectangles, two of which are squares, and adding

ii multiplying length by width of the undivided square (boundary in blue)

a



b



3.2 Factorising

Four different types of factorisation will be considered.

1 Removing the highest common factor (HCF)

Example 8

Factorise $9x^2 + 81x$.

Solution

$$\begin{aligned} 9x^2 + 81x &= 9x \times x + 9x \times 9 \\ &= 9x(x+9) \qquad \text{removing the HCF} \end{aligned}$$

Example 9

Factorise $2a^2 - 8ax^2$.

Solution

$$\begin{aligned} 2a^2 - 8ax^2 &= 2a \times a - 2a \times 4x^2 \\ &= 2a(a - 4x^2) \end{aligned}$$

Example 10

Factorise $7x^2y - 35xy^2$.

Solution

$$7x^2y - 35xy^2 = 7xy(x - 5y) \qquad \text{where } 7xy \text{ is the HCF}$$

2 Grouping of terms

This can be used for expressions containing four terms.

Example 11

Factorise $x^3 + 4x^2 - 3x - 12$.

Solution

The terms in this expression can be grouped as follows:

$$\begin{aligned} x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12) \\ &= x^2(x + 4) - 3(x + 4) \qquad \text{removing the HCF} \\ &= (x^2 - 3)(x + 4) \end{aligned}$$

3 Difference of two squares (DOTS)

$$x^2 - a^2 = (x + a)(x - a)$$

Example 12

Factorise $3x^2 - 75$.

Solution

$$\begin{aligned} 3x^2 - 75 &= 3(x^2 - 25) \qquad \text{3 is the HCF} \\ &= 3(x + 5)(x - 5) \end{aligned}$$

Example 13

Factorise $9x^2 - 36$.

Solution

This is a difference of two squares:

$$\begin{aligned} 9x^2 - 36 &= 9(x^2 - 4) \\ &= 9(x - 2)(x + 2) \end{aligned}$$

Example 14

Factorise $(x - y)^2 - 16y^2$.

Solution

$$\begin{aligned} (x - y)^2 - 16y^2 &= (x - y)^2 - (4y)^2 \\ &= (x - y + 4y)(x - y - 4y) \\ &= (x + 3y)(x - 5y) \end{aligned}$$

4 Factorising quadratic expressions

Example 15

PowerPoint

Factorise $x^2 - 2x - 8$.

Solution

$$x^2 - 2x - 8 = (x + a)(x + b) = x^2 + (a + b)x + ab$$

The values of a and b are such that $ab = -8$

and $a + b = -2$

Values of a and b which satisfy these two conditions are $a = -4$ and $b = 2$

$\therefore\ x^2 - 2x - 8 = (x - 4)(x + 2)$

Example 16

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Factorise $6x^2 - 13x - 15$.

Solution

There are several combinations of factors of $6x^2$ and -15 to consider. Only one combination is correct.

$\therefore$ Factors of $6x^2 - 13x - 15$
$= (6x + 5)(x - 3)$

Factors of $6x^2$	Factors of -15	'Cross-products' add to give $-13x$
$6x$	$+5$	$+5x$
x	-3	$-18x$
		$-13x$

Exercise 3B

1 Factorise each of the following:

a $2x + 4$ **b** $4a - 8$ **c** $6 - 3x$
d $2x - 10$ **e** $18x + 12$ **f** $24 - 16x$

Examples 8, 9

2 Factorise:

a $4x^2 - 2xy$ **b** $8ax + 32xy$ **c** $6ab - 12b$
d $6xy + 14x^2y$ **e** $x^2 + 2x$ **f** $5x^2 - 15x$
g $-4x^2 - 16x$ **h** $7x + 49x^2$ **i** $2x - x^2$

Example 10

j $6x^2 - 9x$ **k** $7x^2y - 6y^2x$ **l** $8x^2y^2 + 6y^2x$

Example 11

3 Factorise:

a $x^3 + 5x^2 + x + 5$ **b** $x^2y^2 - x^2 - y^2 + 1$ **c** $ax + ay + bx + by$
d $a^3 - 3a^2 + a - 3$ **e** $x^3 - bx^2 - a^2x + a^2b$

Examples 12, 13

4 Factorise:

a $x^2 - 36$ **b** $4x^2 - 81$ **c** $2x^2 - 98$

Example 14

d $3ax^2 - 27a$ **e** $(x - 2)^2 - 16$ **f** $25 - (2 + x)^2$
g $3(x + 1)^2 - 12$ **h** $(x - 2)^2 - (x + 3)^2$

Example 15

5 Factorise:

a $x^2 - 7x - 18$	**b** $y^2 - 19y + 48$	**c** $3x^2 - 7x + 2$
d $6x^2 + 7x + 2$	**e** $a^2 - 14a + 24$	**f** $a^2 + 18a + 81$
g $5x^2 + 23x + 12$	**h** $3y^2 - 12y - 36$	**i** $2x^2 - 18x + 28$
j $4x^2 - 36x + 72$	**k** $3x^2 + 15x + 18$	**l** $ax^2 + 7ax + 12a$
m $5x^3 - 16x^2 + 12x$	**n** $48x - 24x^2 + 3x^3$	**o** $(x-1)^2 + 4(x-1) + 3$

Example 16

3.3 Quadratic equations

In this section the solution of quadratic equations by simple factorisation is considered. There are three steps to solving a quadratic equation by factorisation.

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the result that $ab = 0$ implies $a = 0$ or $b = 0$ (or both) (the null factor theorem).

For example

$$\begin{aligned} x^2 - x &= 12 \\ x^2 - x - 12 &= 0 && \text{Step 1} \\ (x-4)(x+3) &= 0 && \text{Step 2} \\ x - 4 = 0 \quad &\text{or} \quad x + 3 = 0 && \text{Step 3} \\ \therefore \quad x = 4 \quad &\text{or} \quad x = -3 \end{aligned}$$

Example 17

Solve $x^2 + 11x + 24 = 0$.

Solution

x^2	$+24$	$+11x$
x	$+3$	$+3x$
x	$+8$	$+8x$
		$+11x$

Factorising we obtain

$$(x+3)(x+8) = 0$$

$$\therefore \quad x + 3 = 0 \quad \text{or} \quad x + 8 = 0$$

$$\therefore \quad x = -3 \quad \text{or} \quad x = -8$$

i.e. both $x = -8$ and $x = -3$ are solutions of $x^2 + 11x + 24 = 0$

To verify, substitute in the equation.

When $x = -8$ $\quad (-8)^2 + 11(-8) + 24 = 0$

$x = -3$ $\quad (-3)^2 + 11(-3) + 24 = 0$

Example 18

Solve $2x^2 + 5x - 12 = 0$.

Solution

$2x^2$	-12	$+5x$
$2x$	-3	$-3x$
x	$+4$	$+8x$
		$+5x$

Factorising gives

$(2x - 3)(x + 4) = 0$

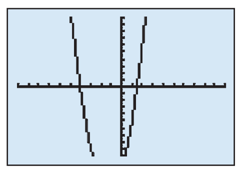
$\therefore \quad 2x - 3 = 0 \quad$ or $\quad x + 4 = 0$

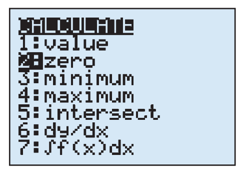
$\therefore \quad x = \frac{3}{2} \quad$ or $\quad x = -4$

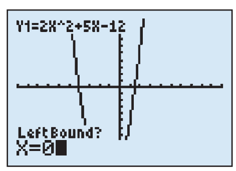
Using a graphics calculator

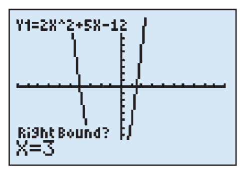
The use of the graphics calculator to solve linear equations was discussed in Chapter 1. Similar techniques can be used to solve quadratic equations. In this case, however, it is advantageous to use **2:zero** from the **CALC** menu. With the graphics calculator exact solutions are not always given, and certainly not if the solutions are not rational.

To solve $2x^2 + 5x - 12 = 0$ first enter $Y1 = 2x^2 + 5x - 12$ in the **Y=** screen. Choose **6:ZStandard** from the **ZOOM** menu to see the graph. It can be seen in this window that there are two values of x which make the value of the expression $2x^2 + 5x - 12$ zero. From the graph they appear to be around $x = 2$ and $x = -4$. The solution to the right will be found. Choose **2:zero** from the **CALC** menu. For the left bound type $x = 0$ (or take the cursor to another suitable point) and press ENTER.

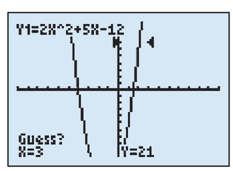


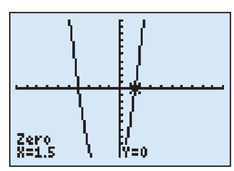




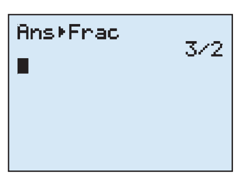


For the right bound choose $x = 3$ and press ENTER. The guess can be 3. Press ENTER twice. To obtain the answer in fractional form press 2ND QUIT to return to the Home screen.





Press 2ND ANS and then choose **1:Frac** from the **MATH** menu. The other solution can be found in a similar manner.



Example 19

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Calculate the length and width of the rectangle.

Solution

Let x cm be the length of the rectangle and y cm the width.

Then $2(x + y) = 20$ and thus $y = 10 - x$.

The area is 24 cm and therefore $x(10 - x) = 24$.

i.e. $\quad 10x - x^2 = 24$

This implies $\quad x^2 - 10x - 24 = 0$

$(x - 6)(x - 4) = 0$

Thus the length is 6 cm or 4 cm. The width is 4 cm or 6 cm.

Exercise 3C

1 Solve each of the following for x:

a $(x - 2)(x - 3) = 0$ **b** $x(2x - 4) = 0$ **c** $(x - 4)(2x - 6) = 0$
d $(3 - x)(x - 4) = 0$ **e** $(2x - 6)(x + 4) = 0$ **f** $2x(x - 1) = 0$
g $(5 - 2x)(6 - x) = 0$ **h** $x^2 = 16$

2 Use a graphics calculator to solve each of the following equations. Give your answer correct to 2 decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 - 4x - 3 = 0$ **c** $-2x^2 - 4x + 3 = 0$

Example 17

3 Solve for x in each of the following:

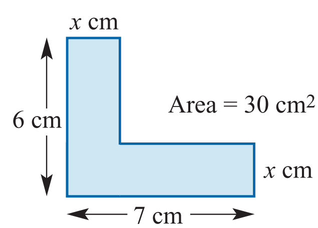
a $x^2 - 6x + 8 = 0$ **b** $x^2 - 8x - 33 = 0$ **c** $x(x + 12) = 64$
d $x^2 + 5x - 14 = 0$ **e** $2x^2 + 5x + 3 = 0$ **f** $4x^2 - 8x + 3 = 0$

Example 18

g $x^2 = 5x + 24$ **h** $6x^2 + 13x + 6 = 0$ **i** $2x^2 - x = 6$
j $6x^2 + 15 = 23x$ **k** $2x^2 - 3x - 9 = 0$ **l** $10x^2 - 11x + 3 = 0$
m $12x^2 + x = 6$ **n** $4x^2 + 1 = 4x$ **o** $x(x + 4) = 5$
p $\frac{1}{7}x^2 = \frac{3}{7}x$ **q** $x^2 + 8x = -15$ **r** $5x^2 = 11x - 2$

4 The bending moment, M, of a simple beam used in bridge construction is given by the formula $M = \frac{wl}{2}x - \frac{w}{2}x^2$.
If $l = 13$ m, $w = 16$ kg/m and $M = 288$ kg m, calculate the value of x.

5 Calculate the value of x.



6 The height, h, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 metres.

7 A polygon with n sides has $\frac{n(n-3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

8 For a particular electric train the tractive 'resistance', R, at speed, v km/h is given by $R = 1.6 + 0.03v + 0.003v^2$. Find v when the tractive resistance is 10.6.

Example 19
9 The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate the length and width of the rectangle.

Example 19
10 The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the altitude.

11 Tickets for a concert are available at two prices. The more expensive ticket is \$30 more than the cheaper one. Find the cost of each type of ticket if a group can buy 10 more of the cheaper tickets than the expensive ones for \$1800.

12 The members of a club hire a bus for \$2100. Seven members withdraw from the club and the remaining members have to pay \$10 more each to cover the cost. How many members originally agreed to go on the bus?

3.4 Graphing quadratics

A quadratic relation is defined by the general rule

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$.

This is called **polynomial** form.

The simplest quadratic relation is $y = x^2$.

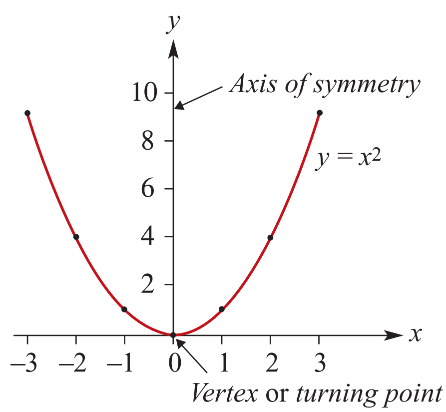
If a table of values is constructed for $y = x^2$ for $-3 \leq x \leq 3$,

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

these points can be plotted and then connected to produce a continuous curve.

Features of the graph of $y = x^2$:

- The graph is called a **parabola**.
- The possible y-values are all positive real numbers and 0. (This is called the **range** of the quadratic and is discussed in a more general context in Chapter 5.)
- It is symmetrical about the y-axis. The line about which the graph is symmetrical is called the **axis of symmetry**.
- The graph has a **vertex** or **turning point** at the origin (0, 0).
- The minimum value of y is 0 and it occurs at the turning point.



By a process called **completing the square** (to be discussed later in this chapter) all quadratics in polynomial form $y = ax^2 + bx + c$ may be transposed into what will be called the **turning point** form:

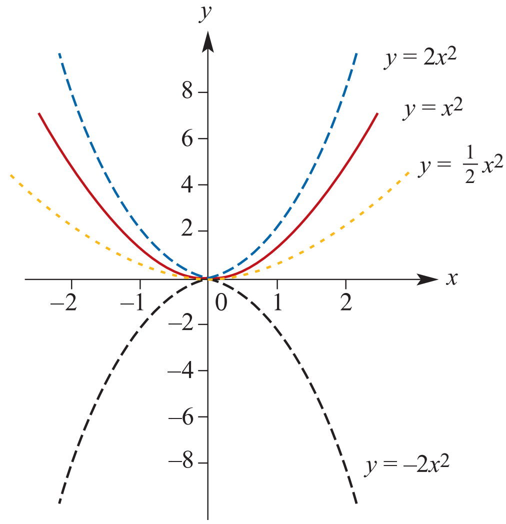
$$y = a(x - h)^2 + k$$

The effect of changing the values of a, h and k on our basic graph of $y = x^2$ will now be investigated. Graphs of the form $y = a(x - h)^2 + k$ are formed by **transforming** the graph of $y = x^2$. A more formal approach to transformations is undertaken in Chapter 5.

i Changing the value of *a*

First consider graphs of the form $y = ax^2$. In this case both $h = 0$ and $k = 0$. In the basic graph of $y = x^2$, a is equal to 1.

The following graphs are shown on the same set of axes.



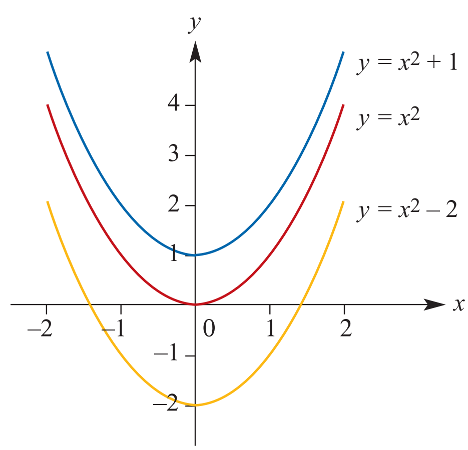
$$y = x^2$$
$$y = 2x^2 \qquad (a = 2)$$
$$y = \frac{1}{2}x^2 \qquad \left(a = \frac{1}{2}\right)$$
$$y = -2x^2 \qquad (a = -2)$$

If $a > 1$ the graph is 'narrower', if $a < 1$ the graph is 'broader'. The transformation which produces the graph of $y = 2x^2$ from the graph of $y = x^2$ is called a **dilation of factor 2 from the *x*-axis**. When a is negative the graph is reflected in the x-axis. The transformation which produces the graph of $y = -x^2$ from the graph of $y = x^2$ is called a **reflection in the *x*-axis**.

ii Changing the value of *k* ($a = 1$ and $h = 0$)

On this set of axes are the graphs of

$$y = x^2$$
$$y = x^2 - 2 \qquad (k = -2)$$
$$y = x^2 + 1 \qquad (k = 1)$$

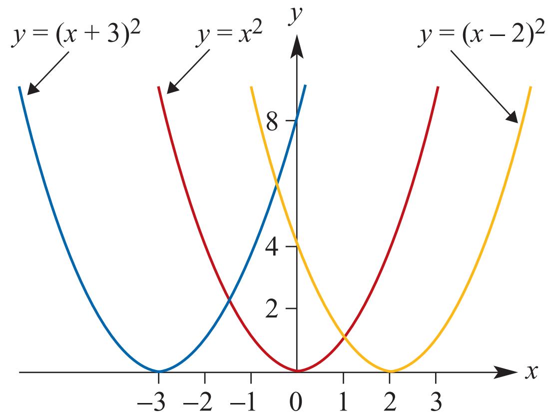


As can be seen, changing k moves the basic graph of $y = x^2$ in a vertical direction. When $k = -2$ the graph is **translated** 2 units in the negative direction of the y-axis. The vertex is now $(0, -2)$ and the range is now all real numbers greater than or equal to -2.
When $k = 1$ the graph is **translated** 1 unit in the positive direction of the y-axis. The vertex is now $(0, 1)$ and the range is now all real numbers greater than or equal to 1.
All other features of the graph are unchanged. The axis of symmetry is still the y-axis.

iii Changing the value of h ($a = 1$ and $k = 0$)

On this set of axes are the graphs of

$$y = x^2$$
$$y = (x - 2)^2 \qquad (h = 2)$$
$$y = (x + 3)^2 \qquad (h = -3)$$



As can be seen, changing h moves the graph in a horizontal direction. When $h = 2$ the graph is **translated** 2 units in the positive direction of the x-axis.

The vertex is now (2, 0) and the axis of symmetry is now the line $x = 2$; however, the range is unchanged and is still all non-negative real numbers. When $h = -3$ the graph is **translated** 3 units in the negative direction of the x-axis. The vertex is now $(-3, 0)$ and the axis of symmetry is now the line $x = -3$; however, again the range is unchanged and is still all non-negative real numbers.

All these effects can be combined and the graph of any quadratic, expressed in the form $y = a(x - h)^2 + k$, can be sketched.

Example 20

Sketch the graph of $y = x^2 - 3$.

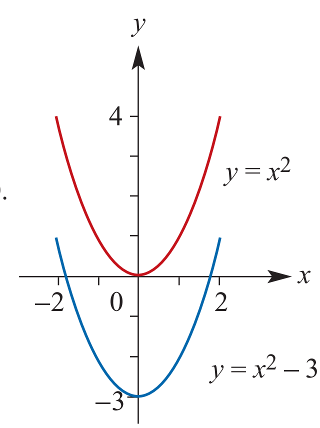
Solution

The graph of $y = x^2 - 3$ is obtained from the graph of $y = x^2$ by a translation of 3 units in the negative direction of the x-axis

The vertex is now at $(0, -3)$.

The axis of symmetry is the line with equation $x = 0$.

The x-axis intercepts are determined by solving the equation



$$x^2 - 3 = 0$$
$$\therefore \quad x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

$\therefore$ x-axis intercepts are $\pm\sqrt{3}$.

Example 21

Sketch the graph of $y = -(x + 1)^2$.

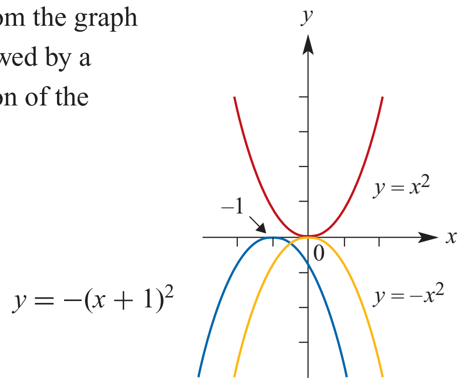
Solution

The graph of $y = -(x + 1)^2$ is obtained from the graph of $y = x^2$ by a reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis.

The vertex is now at $(-1, 0)$.

The axis of symmetry is the line with equation $x = -1$.

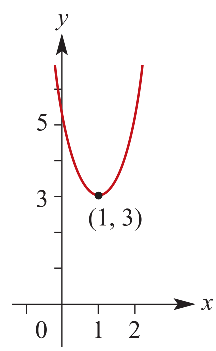
The x-axis intercept is $(-1, 0)$.



Example 22

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution



The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range will be $y \geq 3$.

To add further detail to our graph, the y-axis intercept and the x-axis intercepts (if any) can be found.

y-axis intercept: Let $x = 0$ $\quad y = 2(0 - 1)^2 + 3$

$y = 5$

x-axis intercept(s): In this example the minimum value of y is 3, so y cannot be 0.

$\therefore$ this graph has no x-axis intercepts.

Note: Let $y = 0$ and try to solve for x.

$$\begin{aligned} 0 &= 2(x - 1)^2 + 3 \\ -3 &= 2(x - 1)^2 \\ -\frac{3}{2} &= (x - 1)^2 \end{aligned}$$

As the square root of a negative number is not a real number, this equation has no real solutions.

Example 23

Sketch the graph of $y = -(x + 1)^2 + 4$.

Solution

The vertex has coordinates $(-1, 4)$.

The axis of symmetry is the line $x = -1$.

***y*-axis intercept**:

Let $x = 0$ $\quad y = -(0 + 1)^2 + 4$

$y = 3$

***x*-axis intercepts**:

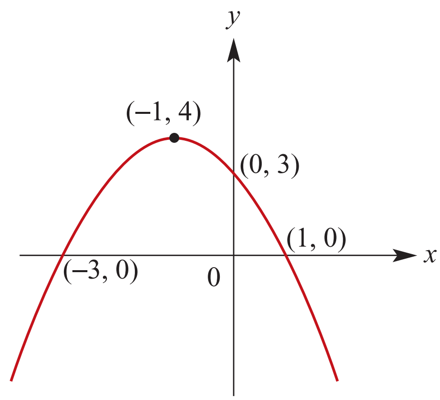
Let $y = 0$ $\quad 0 = -(x + 1)^2 + 4$

$(x + 1)^2 = 4$

$x + 1 = \pm 2$

$x = \pm 2 - 1$

$\therefore$ x-axis intercepts are $(1, 0)$ and $(-3, 0)$



Exercise 3D

Find: **i** the coordinates of the turning point **ii** the axis of symmetry

iii the x-axis intercepts (if any)

of each case and use this information to help sketch the following graphs.

Example 20 **a** $y = x^2 - 4$ **b** $y = x^2 + 2$ **c** $y = -x^2 + 3$

Example 21 **d** $y = -2x^2 + 5$ **e** $y = (x - 2)^2$ **f** $y = (x + 3)^2$

Example 22 **g** $y = -(x + 1)^2$ **h** $y = -\frac{1}{2}(x - 4)^2$ **i** $y = (x - 2)^2 - 1$

Example 23 **j** $y = (x - 1)^2 + 2$ **k** $y = (x + 1)^2 - 1$ **l** $y = -(x - 3)^2 + 1$

m $y = (x + 2)^2 - 4$ **n** $y = 2(x + 2)^2 - 18$ **o** $y = -3(x - 4)^2 + 3$

p $y = -\frac{1}{2}(x + 5)^2 - 2$ **q** $y = 3(x + 2)^2 - 12$ **r** $y = -4(x - 2)^2 + 8$

3.5 Completing the square

In order to use the above technique for sketching quadratics, it is necessary for the quadratic to be expressed in **turning point** form.

To transpose a quadratic in **polynomial** form we must **complete the square**.

Consider the perfect square $\quad (x + a)^2$

which when expanded becomes $\quad x^2 + 2ax + a^2$

The last term of the expansion is the square of half the coefficient of the middle term.

Now consider the quadratic $\quad y = x^2 + 2x - 3$

This is not a perfect square. We can however find the 'correct' last term to make this a perfect square.

If the last term is $1 = \left(\frac{1}{2} \times 2\right)^2$, then

$$\begin{aligned} y &= x^2 + 2x + 1 \\ &= (x+1)^2 \text{ a perfect square.} \end{aligned}$$

In order to keep our original quadratic 'intact', we both add and subtract the 'correct' last term. For example:

$$y = x^2 + 2x - 3$$

becomes $\quad y = (x^2 + 2x + 1) - 1 - 3$

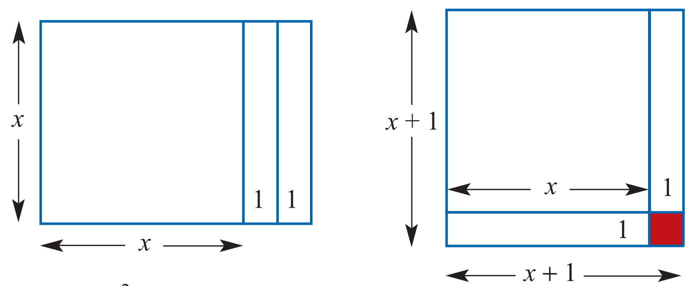
This can now be simplified to

$$y = (x+1)^2 - 4$$

Hence the **vertex** (turning point) can now be seen to be the point with coordinates $(-1, -4)$.

In the above example the coefficient of x^2 was 1. If the coefficient is not 1, this coefficient must first be 'factored out' before proceeding to complete the square.

Completing the square for $x^2 + 2x$ is represented in the following diagram. The diagram to the left shows $x^2 + 2x$. The small rectangle to the right is moved to the 'base' of the x by x square. The red square of area 1 unit is added. Thus $x^2 + 2x + 1 = (x+1)^2$.



The process of completing the square can also be used for the solution of equations.

Example 24

Solve each of the following equations for x by first completing the square:

a $x^2 - 3x + 1 = 0$ **b** $x^2 - 2kx + 1 = 0$ **c** $2x^2 - 3x - 1 = 0$

Solution

a $x^2 - 3x + 1 = 0$

Completing the square:

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

Therefore $\quad \left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$

and $x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

Hence $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$.

b $x^2 - 2kx + 1 = 0$

Completing the square:

$x^2 - 2kx + k^2 + 1 - k^2 = 0$

$(x-k)^2 = k^2 - 1$

Therefore $x - k = \pm\sqrt{k^2 - 1}$

And $x = k \pm \sqrt{k^2 - 1}$

Note: If $k = \pm 1$ then $x = \pm 1$.

If $k > 1$ or $k < -1$ then there are two solutions.

If $-1 < k < 1$ then there are no solutions.

c $2x^2 - 3x - 1 = 0$

$$2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$$

$$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \frac{1}{2} - \left(\frac{3}{4}\right)^2 = 0$$

Divide both sides by 2 and then complete the square.

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Therefore $x - \dfrac{3}{4} = \pm\dfrac{\sqrt{17}}{4}$

and $x = \dfrac{3}{4} \pm \dfrac{\sqrt{17}}{4} = \dfrac{3 \pm \sqrt{17}}{4}$

Example 25

Find the coordinates of the vertex by completing the square and hence sketch the graph of $y = -2x^2 + 6x - 8$.

Solution

$$y = -2x^2 + 6x - 8$$

$$y = -2(x^2 - 3x + 4)$$

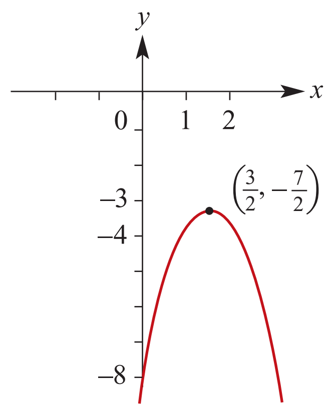
-2 is 'factored out'

$$y = -2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4\right)$$

$$y = -2\left\{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4\right\}$$

$$y = -2\left\{\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}\right\}$$

$$y = -2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$$



$\therefore$ The vertex is $\left(\dfrac{3}{2}, -\dfrac{7}{2}\right)$.

y-axis intercept $= -8$.

Graph has maximum value of $-\dfrac{7}{2}$, $\therefore$ there are no x-axis intercepts.

The axis of symmetry is $x = \dfrac{3}{2}$.

Exercise 3E

1 Expand each of the following:

a $(x - 1)^2$ **b** $(x + 2)^2$ **c** $(x - 3)^2$ **d** $(-x + 3)^2$

e $(-x - 2)^2$ **f** $(x - 5)^2$ **g** $\left(x - \dfrac{1}{2}\right)^2$ **h** $\left(x - \dfrac{3}{2}\right)^2$

2 Factorise each of the following:

a $x^2 - 4x + 4$ **b** $x^2 - 12x + 36$ **c** $-x^2 + 4x - 4$

d $2x^2 - 8x + 8$ **e** $-2x^2 + 12x - 18$ **f** $x^2 - x + \frac{1}{4}$

g $x^2 - 3x + \frac{9}{4}$ **h** $x^2 + 5x + \frac{25}{4}$

Example 24 **3** Solve each of the following equations for x by first completing the square:

a $x^2 - 2x - 1 = 0$ **b** $x^2 - 4x - 2 = 0$ **c** $x^2 - 6x + 2 = 0$

d $x^2 - 5x + 2 = 0$ **e** $2x^2 - 4x + 1 = 0$ **f** $3x^2 - 5x - 2 = 0$

g $x^2 + 2x + k = 0$ **h** $kx^2 + 2x + k = 0$ **i** $x^2 - 3kx + 1 = 0$

Example 25 **4** Express each of the following in the form $y = a(x - h)^2 + k$.
Hence state the coordinates of the turning point and sketch the graph in each case.

a $x^2 - 2x + 3$ **b** $x^2 + 4x + 1$ **c** $x^2 - 3x + 1$

d $x^2 - 8x + 12$ **e** $x^2 - x - 2$ **f** $2x^2 + 4x - 2$

g $-x^2 + 4x + 1$ **h** $-2x^2 - 12x - 12$ **i** $3x^2 - 6x + 12$

3.6 Sketching quadratics in polynomial form

It is not essential to convert quadratics to turning point form in order to sketch the graph. We can find the x- and y-axis intercepts and the axis of symmetry from polynomial form by alternative methods and use these details to sketch the graph.

***y*-axis intercept:** Letting $x = 0$ in the general equation $y = ax^2 + bx + c$ we have

$$y = a(0)^2 + b(0) + c$$
$$y = c$$

i.e. the y-axis intercept is always equal to c.

***x*-axis intercepts:** Letting $y = 0$ in the general equation we have

$$0 = ax^2 + bx + c$$

In order to solve such an equation it is necessary to factorise the right-hand side and use the **null factor theorem**.

Once the x-axis intercepts have been found, the turning point can be found by using the symmetry properties of the parabola.

Example 26

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 4x$.

Solution

Step 1 $c = 0$, $\therefore$ y-axis intercept is $(0, 0)$

Step 2 Set $y = 0$ and factorise right-hand side of equation:

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$

$\therefore x = 0$ or $x = 4$

x-axis intercepts are $(0, 0)$ and $(4, 0)$

Step 3 Axis of symmetry is the line with

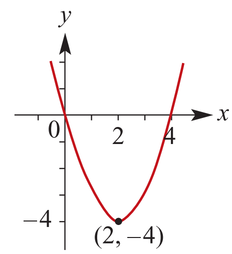
equation $x = \dfrac{0+4}{2}$

i.e. $x = 2$

When $x = 2$, $y = (2)^2 - 4(2)$

$= -4$

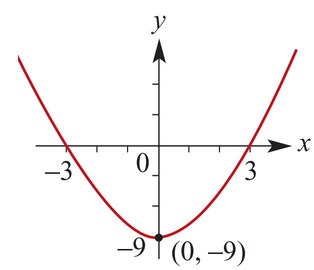
$\therefore$ turning point has coordinates $(2, -4)$.



Example 27

Find the x- and y-axis intercepts and the coordinates of the turning point, and hence sketch the graph of $y = x^2 - 9$.

Solution



Step 1 $c = -9$, $\therefore$ y-axis intercept is $(0, -9)$.

Step 2 Set $y = 0$ and factorise right-hand side:

$$0 = x^2 - 9$$
$$0 = (x + 3)(x - 3)$$

$\therefore x = -3$ or $x = 3$

x-axis intercepts are $(-3, 0)$ and $(3, 0)$

Step 3 Axis of symmetry is the line with equation $x = \dfrac{-3+3}{2}$

i.e. $x = 0$

When $x = 0$, $y = (0)^2 - 9$

$= -9$

$\therefore$ turning point has coordinates $(0, -9)$.

Example 28

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

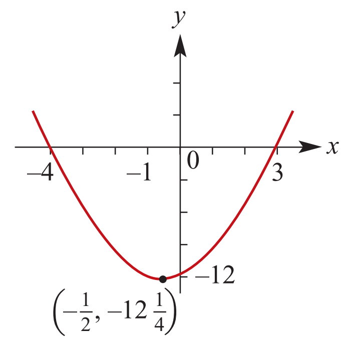
Step 1 $c = -12$, $\therefore$ y-axis intercept is $(0, -12)$

Step 2 Set $y = 0$ and factorise the right-hand side:

$$0 = x^2 + x - 12$$
$$0 = (x + 4)(x - 3)$$

$\therefore x = -4$ or $x = 3$

x-axis intercepts are $(-4, 0)$ and $(3, 0)$



Step 3 Due to the symmetry of the parabola, the axis of symmetry will be the line bisecting the two x-axis intercepts.

$\therefore$ the axis of symmetry is the line with equation $x = \dfrac{-4 + 3}{2} = -\dfrac{1}{2}$.

$$\text{When } x = -\frac{1}{2}, y = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 12$$
$$= -12\frac{1}{4}$$

$\therefore$ the turning point has coordinates $\left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$.

Exercise 3F

1 **a** A parabola has x-axis intercepts 4 and 10. State the x-coordinate of the vertex.
b A parabola has x-axis intercepts 6 and 8. State the x-coordinate of the vertex.
c A parabola has x-axis intercepts -6 and 8. State the x-coordinate of the vertex.

2 **a** A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at 6. Find the other x-axis intercept.
b A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at -4. Find the other x-axis intercept.
c A parabola has vertex $(-2, 6)$ and one of the x-axis intercepts is at the origin. Find the other x-axis intercept.

3 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

Examples 26, 27

a $y = x^2 - 1$ **b** $y = x^2 + 6x$ **c** $y = 25 - x^2$ **d** $y = x^2 - 4$
e $y = 2x^2 + 3x$ **f** $y = 2x^2 - 4x$ **g** $y = -2x^2 - 3x$ **h** $y = x^2 + 1$

4 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

Example 28

a $y = x^2 + 3x - 10$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 + 2x - 3$
d $y = x^2 + 4x + 3$ **e** $y = 2x^2 - x - 1$ **f** $y = 6 - x - x^2$
g $y = -x^2 - 5x - 6$ **h** $y = x^2 - 5x - 24$

3.7 The general quadratic formula

Not all quadratics can be factorised by inspection and it is often difficult to find the x-axis intercepts this way. If the general expression for a quadratic in polynomial form is considered, a general formula can be developed by using the 'completing the square' technique. This can be used to solve quadratic equations.

To solve $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide all terms by a.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

Complete the square by adding and subtracting $\left(\frac{b}{2a}\right)^2$.

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Note: The use of the **quadratic formula** is an alternative method to 'completing the square' for solving quadratic equations, but is probably not as useful as 'completing the square' for curve sketching, which gives the turning point coordinates directly.

It should be noted that the equation of the axis of symmetry can be derived from this general formula.

The axis of symmetry is the line with equation $x = -\frac{b}{2a}$.

Also, from the formula it can be seen that:

if $b^2 - 4ac > 0$ there are two solutions

if $b^2 - 4ac = 0$ there is one solution

if $b^2 - 4ac < 0$ there are no real solutions.

This will be further explored in the next section.

Example 29

Solve each of the following equations for x by using the quadratic formula:

a $x^2 - x - 1 = 0$ **b** $x^2 - 2kx - 3 = 0$

Solution

a $x^2 - x - 1 = 0$

$a = 1$, $b = -1$ and $c = -1$

The formula gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

b $2x^2 - 2kx - 3 = 0$

$a = 2$, $b = -2k$ and $c = -3$

The formula gives

$$x = \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4 \times 2 \times (-3)}}{2 \times 2}$$

$$= \frac{2k \pm \sqrt{4k^2 + 24}}{4}$$

$$= \frac{k \pm \sqrt{k^2 + 6}}{2}$$

Example 30

Sketch the graph of $y = -3x^2 - 12x - 7$ by first using the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$ the y-axis intercept is $(0, -7)$.

Turning point coordinates

Axis of symmetry: $x = -\dfrac{b}{2a}$

$$= -\frac{(-12)}{2 \times -3}$$

$$= -2$$

When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7$

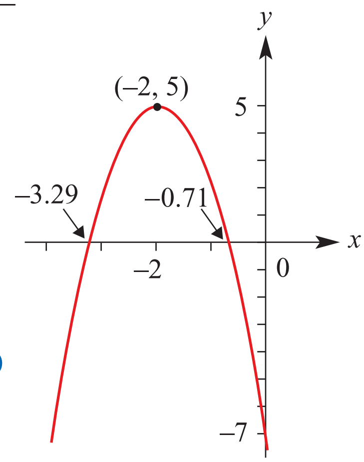
$$= 5$$

$\therefore$ turning point coordinates are $(-2, 5)$.

x-axis intercepts:

$$-3x^2 - 12x - 7 = 0$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)} \\
&= \frac{12 \pm \sqrt{60}}{-6} \\
&= \frac{12 \pm 2\sqrt{15}}{-6} \\
&= \frac{6 \pm \sqrt{15}}{-3} \\
&= \frac{6 \pm 3.87}{-3} \quad \text{(to 2 decimal places)} \\
&= -3.29 \text{ or } -0.71
\end{aligned}$$



Exercise 3G

1 For each of the following the coefficients a, b and c of a quadratic $y = ax^2 + bx + c$ are given. Find:

i $b^2 - 4ac$ **ii** $\sqrt{b^2 - 4ac}$ in simplest surd form

a $a = 2, b = 4$ and $c = -3$ **b** $a = 1, b = 10$ and $c = 18$
c $a = 1, b = 10$ and $c = -18$ **d** $a = -1, b = 6$ and $c = 15$
e $a = 1, b = 9$ and $c = -27$

2 Simplify each of the following:

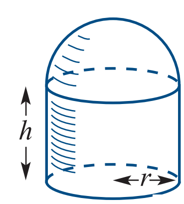
a $\dfrac{2 + 2\sqrt{5}}{2}$ **b** $\dfrac{9 - 3\sqrt{5}}{6}$ **c** $\dfrac{5 + 5\sqrt{5}}{10}$ **d** $\dfrac{6 + 12\sqrt{2}}{6}$

Examples 29, 30

3 Solve each of the following for x. Give exact answers.

a $x^2 + 6x = 4$ **b** $x^2 - 7x - 3 = 0$ **c** $2x^2 - 5x + 2 = 0$
d $2x^2 + 4x - 7 = 0$ **e** $2x^2 + 8x = 1$ **f** $5x^2 - 10x = 1$
g $-2x^2 + 4x - 1 = 0$ **h** $2x^2 + x = 3$ **i** $2.5x^2 + 3x + 0.3 = 0$
j $-0.6x^2 - 1.3x = 0.1$ **k** $2kx^2 - 4x + k = 0$ **l** $2(1 - k)x^2 - 4kx + k = 0$

4 The surface area, S, of a cylindrical tank with a hemispherical top is given by the formula $S = ar^2 + brh$, where $a = 9.42$ and $b = 6.28$. What is the radius of a tank of height 6 m which has a surface area of 125.6 m^2?



Example 29

5 Sketch the graphs of the following parabolas. Use the quadratic formula to find the x-axis intercepts (if they exist) and the axis of symmetry and, hence, the turning point.

a $y = x^2 + 5x - 1$ **b** $y = 2x^2 - 3x - 1$ **c** $y = -x^2 - 3x + 1$

d $y + 4 = x^2 + 2x$ **e** $y = 4x^2 + 5x + 1$ **f** $y = -3x^2 + 4x - 2$

3.8 Iteration

Quadratic equations can be solved using a process of **simple iteration**. In this section the process is discussed. Knowledge of sequence notation from General Mathematics is useful but not essential for this topic.

Consider the quadratic equation $x^2 + 3x - 5 = 0$.

This can be rearranged to the equivalent equations

$$x(x+3) = 5 \quad \text{and} \quad x = \frac{5}{x+3}$$

Solving the equation $x^2 + 3x - 5 = 0$ is equivalent to solving the simultaneous equations

$$y = x \quad \text{and} \quad y = \frac{5}{x+3}$$

The equation $x = \dfrac{5}{x+3}$ can be used to form a difference equation (**iterative equation**)

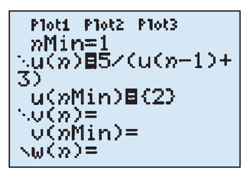
$$x_{n+1} = \frac{5}{x_n + 3} \quad \text{or} \quad x_n = \frac{5}{x_{n-1} + 3}$$

The elements of the sequence are called **iterations**.

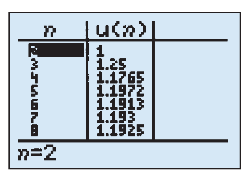
Using a graphics calculator

Choose a starting value near the positive solution of $x^2 + 3x - 5 = 0$. Let $x_1 = 2$.

Enter this iterative equation in the graphics calculator as shown. Note that **Seq** has been chosen from the **MODE** menu.



Press 2ND and TABLE to obtain a table of values.



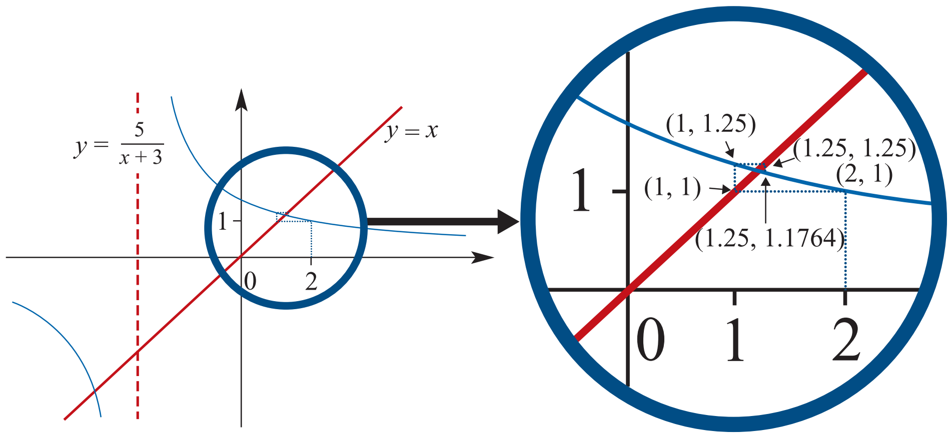
$x_1 = 2, x_2 = 1, x_3 = 1.25, x_4 = 1.1764\ldots,$

$x_{15} = 1.19258241\ldots, x_{16} = 1.19258239\ldots$

The sequence converges to $\dfrac{-3 + \sqrt{29}}{2}$.

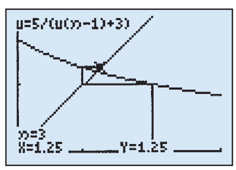
The convergence can be illustrated with a **web** diagram.

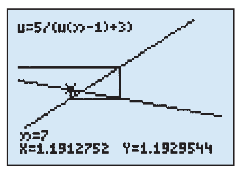
The graphs of $y = \dfrac{5}{x+3}$ and $y = x$ are sketched on the one set of axes and the 'path' of the sequence is illustrated.



This can be seen on a graphics calculator. The **Seq** mode has been chosen and **Web** selected from the **FORMAT** menu. Set the **WINDOW** with **nMin** $= 1$, **nMax** $= 7$, **Xmin** $= 0$, **Xmax** $= 3$, **Ymin** $= 0$ and **Ymax** $= 2$.

Press $\boxed{\text{TRACE}}$ and use the arrow ▶ to move through the sequence. Use **ZBox** from the **ZOOM** menu so that iterations up to $n = 7$ can be seen.





Try other starting points, for example $x_1 = -400$ or $x_1 = 200$.

Convergence is always towards the solution $x = \dfrac{-3 + \sqrt{29}}{2}$.

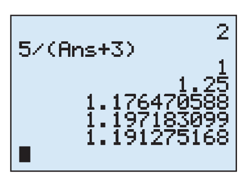
Other arrangements of the equation $x^2 + 3x - 5 = 0$ can be tried to find the other solution. For example:

A $\quad x = \dfrac{-x^2 + 5}{3}$ yields the sequence $x_{n+1} = \dfrac{-(x_n)^2 + 5}{3}$. This converges to $\dfrac{-3 + \sqrt{29}}{2}$ again if $x_1 = 2$ is used for the start. $x_1 = -9$ yields no convergence.

B $\quad x = \dfrac{5}{x} - 3$ yields the sequence $x_{n+1} = \dfrac{5}{x_n} - 3$. This converges to $\dfrac{-3 - \sqrt{29}}{2}$ for the start $x_1 = -9$.

C $\quad x = -\sqrt{5 - 3x}$ yields the sequence $x_{n+1} = -\sqrt{5 - 3x_n}$. This converges to $\dfrac{-3 - \sqrt{29}}{2}$ for the start $x_1 = -10$.

Another method which can be used to generate the sequence is to use **ANS** in the Home screen. Thus for $x_{n+1} = \dfrac{5}{x_n + 3}$ first enter 2 and press $\boxed{\text{ENTER}}$ and then enter 5/(ANS + 3). Press $\boxed{\text{ENTER}}$ repeatedly to generate the sequence.



Exercise 3H

Solve the following equations by using iteration:

a $x^2 + 5x - 10 = 0$ using $x_{n+1} = \dfrac{10}{x_n + 5}$ **b** $x^2 - 3x - 5 = 0$ using $x_{n+1} = \dfrac{5}{x_n - 3}$

c $x^2 + 2x - 7 = 0$ using $x_{n+1} = \dfrac{7}{x_n + 2}$ **d** $-x^2 - 2x + 5 = 0$ using $x_{n+1} = \dfrac{5}{x_n + 2}$

3.9 The discriminant

When graphing quadratics it is apparent that the number of x-axis intercepts a parabola may have is:

i zero – the graph is either all above or all below the x-axis

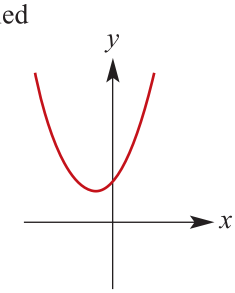
ii one – the graph touches the x-axis; the turning point is the x-axis intercept

or **iii** two – the graph crosses the x-axis.

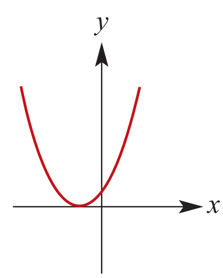
By considering the formula for the general solution to a quadratic equation, $ax^2 + bx + c = 0$, we can establish whether a parabola will have zero, one or two x-axis intercepts.

The expression $b^2 - 4ac$, which is part of the quadratic formula, is called the **discriminant**.

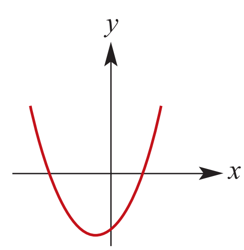
i If the discriminant $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has zero solutions and the corresponding parabola will have no x-axis intercepts.



ii If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



iii If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



Example 31

Find the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

Note: Discriminant is denoted by the symbol $\triangle$.

a Discriminant $\triangle = b^2 - 4ac$

$$= (-6)^2 - (4 \times 1 \times 8)$$
$$= 4$$

As $\triangle > 0$ the graph intersects the x-axis at two distinct points, i.e. there are two distinct solutions of the equation $x^2 - 6x + 8 = 0$.

b $\triangle = b^2 - 4ac$

$$= (-8)^2 - (4 \times 1 \times 16)$$
$$= 64 - 64$$
$$= 0$$

As $\triangle = 0$ the graph touches the x-axis, i.e. there is one solution of the equation $x^2 - 8x + 16 = 0$.

c $\triangle = b^2 - 4ac$

$$= (-3)^2 - (4 \times 2 \times 4)$$
$$= -23$$

As $\triangle < 0$ the graph does not intersect the x-axis, i.e. there are no real solutions for the equation $2x^2 - 3x + 4 = 0$.

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation.

For a, b and c rational:

- If $\triangle = b^2 - 4ac$ is a perfect square, which is not zero, then the quadratic equation has two rational solutions.
- If $\triangle = b^2 - 4ac = 0$ the quadratic equation has one rational solution.
- If $\triangle = b^2 - 4ac$ is not a perfect square then the quadratic equation has two irrational solutions.

Exercise 3I

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 + 3x - 4$

d $2x^2 + 3x - 4$ **e** $-2x^2 + 3x + 4$

Example 31

2 Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis:

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$
d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - x - 1$

3 By examining the discriminant, find the number of roots of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $10x^2 - x - 3 = 0$
d $2x^2 + 8x - 7 = 0$ **e** $3x^2 - 8x - 7 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of roots for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$
d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 2 = 0$

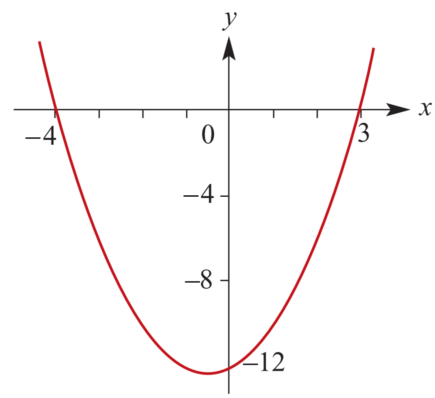
5 Find the discriminant for the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number and hence show that the equation has rational solution(s).

Exercise 3J contains more exercises involving the discriminant.

3.10 Solving quadratic inequations

Example 32

Solve $x^2 + x - 12 > 0$.



Solution

Step 1 Solve the equation $x^2 + x - 12 = 0$

$$(x + 4)(x - 3) = 0$$

$\therefore\ x = 3$ or $x = -4$

Step 2 Sketch the graph of the quadratic $y = x^2 + x - 12$.

Step 3 Use the graph to determine the values of x for which $x^2 + x - 12 > 0$.
From the graph it can be seen that $x^2 + x - 12 > 0$ when $x < -4$ or when $x > 3$.
$\therefore \{x : x^2 + x - 12 > 0\} = \{x : x < -4\} \cup \{x : x > 3\}$

Example 33

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a 1 solution **b** no solution **c** 2 distinct solutions

Solution

For the quadratic $3x^2 - 2mx + 3$, $\triangle = 4m^2 - 36$.

a For 1 solution

$$\begin{aligned} \triangle &= 0 \\ \therefore 4m^2 - 36 &= 0 \\ \therefore \quad m^2 &= 9 \\ m &= \pm 3 \end{aligned}$$

b For no solution

$$\triangle < 0$$

i.e. $4m^2 - 36 < 0$

From the graph

$$-3 < m < 3$$

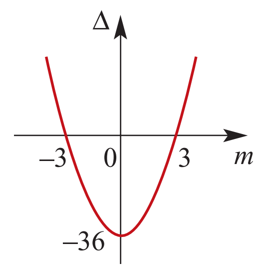
c For two distinct solutions

$$\triangle > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$m > 3$ or $m < -3$



Example 34

Show that the solutions of the equation $3x^2 + (m - 3)\,x - m$ are rational for all rational values of m.

Solution

$$\begin{aligned} \triangle &= (m-3)^2 - 4 \times 3 \times (-m) \\ &= m^2 - 6m + 9 + 12m \\ &= m^2 + 6m + 9 \\ &= (m+3)^2 \geq 0 \text{ for all } m \end{aligned}$$

Furthermore $\triangle$ is a perfect square for all m.

Exercise 3J

Example 32

1 Solve each of the following inequalities:

a $x^2 + 2x - 8 \geq 0$ **b** $x^2 - 5x - 24 < 0$ **c** $x^2 - 4x - 12 \leq 0$

d $2x^2 - 3x - 9 > 0$ **e** $6x^2 + 13x < -6$ **f** $-x^2 - 5x - 6 \geq 0$

g $12x^2 + x > 6$ **h** $10x^2 - 11x \leq -3$ **i** $x(x - 1) \leq 20$

j $4 + 5p - p^2 \geq 0$ **k** $3 + 2y - y^2 < 0$ **l** $x^2 + 3x \geq -2$

Example 33

2 Find the values of m for which each of the following equations:

i has no solutions **ii** has one solution **iii** has two distinct solutions

a $x^2 - 4mx + 20 = 0$ **b** $mx^2 - 3mx + 3 = 0$

c $5x^2 - 5mx - m = 0$ **d** $x^2 + 4mx - 4(m - 2) = 0$

3 Find the values of p for which the equation $px^2 + 2(p + 2)x + p + 7 = 0$ has no real solution.

4 Find the values of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has one solution.

5 Find the values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis.

6 Show that the equation $(p^2 + 1)x^2 + 2pqx + q^2 = 0$ has no real solution for any values of p and q $(q \neq 0)$.

Example 34 7 For m and n rational numbers show that $mx^2 + (2m + n)x + 2n = 0$ has rational solutions.

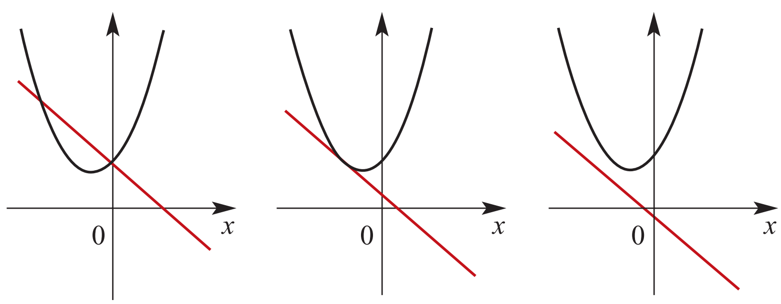
3.11 Solving simultaneous linear and quadratic equations

Excel

As discussed in Section 1.3, when solving simultaneous linear equations we are actually finding the point of intersection of the two linear graphs involved.

If we wish to find the point or points of intersection between a straight line and a parabola we can solve the equations simultaneously.

It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola we may get two, one or zero points of intersection.



Two points of intersection One point of intersection No point of intersection

If there is one point of intersection between the parabola and the straight line then the line is a **tangent** to the parabola.

As we usually have the quadratic equation written with y as the subject, it is necessary to have the linear equation written with y as the subject (i.e. in gradient form).

Then the linear expression for y can be substituted into the quadratic equation.

CAS APPENDIX

Example 35

Find the points of intersection of the line with the equation $y = -2x + 4$ and the parabola with the equation $y = x^2 - 8x + 12$.

Solution

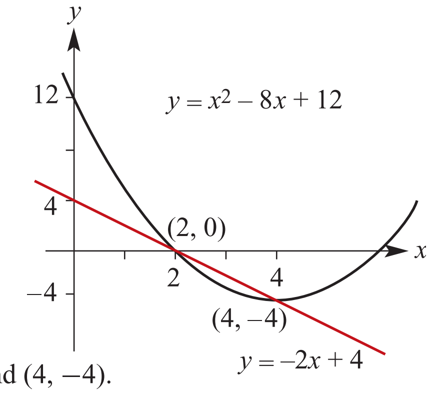
At the point of intersection

$$x^2 - 8x + 12 = -2x + 4$$
$$x^2 - 6x - 8 = 0$$
$$(x - 2)(x - 4) = 0$$

Hence $x = 2$ or $x = 4$

When $x = 2, y = -2(2) + 4 = 0$

$x = 4, y = -2(4) + 4 = -4$



Therefore the points of intersection are (2, 0) and (4, –4).

The result can be shown graphically.

Example 36

Prove that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

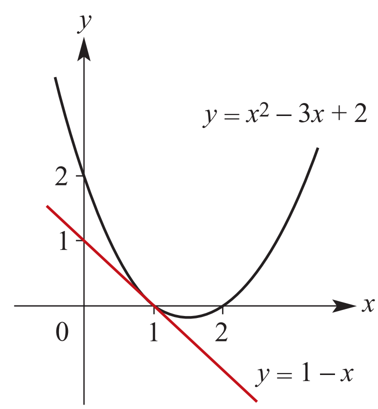
Solution

At the point of intersection:

$$x^2 - 3x + 2 = 1 - x$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$

Therefore $x = 1$

When $x = 1, y = 0$

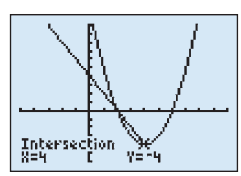


The straight line just touches the parabola.

This can be shown graphically.

Using a graphics calculator

Simultaneous equations involving a linear equation and a quadratic equation can be solved using **5:intersect** from the **CALC** menu. The result for one of the points of intersection of the graphs in Example 35 is shown here.



Exercise 3K

Example 35

1 Solve each of the following pairs of equations:

a $y = x^2 + 2x - 8$
$y = 2 - x$

b $y = x^2 - x - 3$
$y = 4x - 7$

c $y = x^2 + x - 5$
$y = -x - 2$

d $y = x^2 + 6x + 6$
$y = 2x + 3$

e $y = 6 - x - x^2$
$y = -2x - 2$

f $y = x^2 + x + 6$
$y = 6x + 8$

Example 36

2 Prove that, for the pairs of equations given, the straight line meets the parabola only once:

a $y = x^2 - 6x + 8$
$y = -2x + 4$

b $y = x^2 - 2x + 6$
$y = 4x - 3$

c $y = 2x^2 + 11x + 10$
$y = 3x + 2$

d $y = x^2 + 7x + 4$
$y = -x - 12$

3 Solve each of the following pairs of equations:

a $y = x^2 - 6x$
$y = 8 + x$

b $y = 3x^2 + 9x$
$y = 32 - x$

c $y = 5x^2 + 9x$
$y = 12 - 2x$

d $y = -3x^2 + 32x$
$y = 32 - 3x$

e $y = 2x^2 - 12$
$y = 3(x - 4)$

f $y = 11x^2$
$y = 21 - 6x$

4 **a** Find the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$. (Hint: Consider the discriminant of the resulting quadratic.)

b **i** Sketch the parabola with equation $y = -2x^2 - 6x + 2$.

ii Find the values of m for which the straight line $y = mx + 6$ is tangent to the parabola. (Hint: Use the discriminant of the resulting quadratic.)

5 **a** Find the value of c such that the line with equation $y = 2x + c$ is tangent to the parabola with equation $y = x^2 + 3x$.

b Find the possible values of c such that the line with equation $y = 2x + c$ twice intersects the parabola with equation $y = x^2 + 3x$.

6 Find the value(s) of a such that the line with equation $y = x$ is tangent to the parabola with equation $y = x^2 + ax + 1$.

7 Find the value of b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

8 Find the equation of the straight line(s) which pass through the point $(1, -2)$ and is (are) tangent to the parabola with equation $y = x^2$.

3.12 Determining quadratic rules

It is possible to find the quadratic rule to fit given points, if it is assumed that the points lie on a suitable parabola.

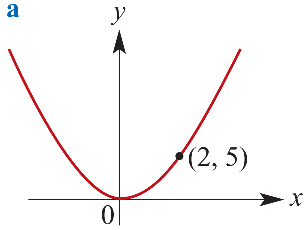
Example 37

Excel

Determine the quadratic rule for each of the following graphs, assuming each is a parabola.

Solution

a



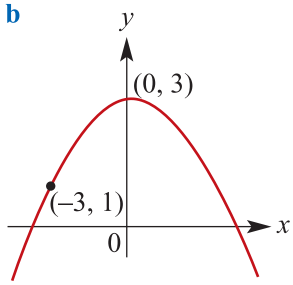
a This is of the form $y = ax^2$

When $x = 2$, $y = 5$, thus $5 = 4a$

Therefore $a = \frac{5}{4}$

and the rule is $y = \frac{5}{4}x^2$

b



b This is of the form $y = ax^2 + c$

For $(0, 3)$ $\quad 3 = a(0) + c$

Therefore $c = 3$

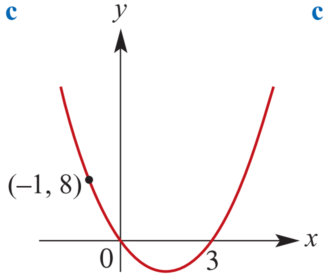
For $(-3, 1)$ $\quad 1 = a(-3)^2 + 3$

$1 = 9a + 3$

Therefore $a = -\frac{2}{9}$

and the rule is $y = -\frac{2}{9}x^2 + 3$

c



c This is of the form $y = ax(x - 3)$

For $(-1, 8)$ $\quad 8 = -a(-1 - 3)$

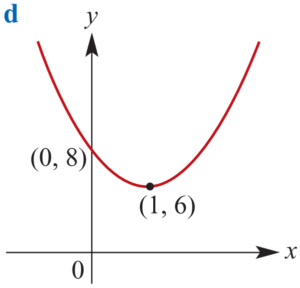
$8 = 4a$

Therefore $a = 2$

and the rule is $y = 2x(x - 3)$

$y = 2x^2 - 6x$

d



d This is of the form $y = k(x - 1)^2 + 6$

When $x = 0$, $\quad y = 8$

$\therefore$ $\quad 8 = k + 6$

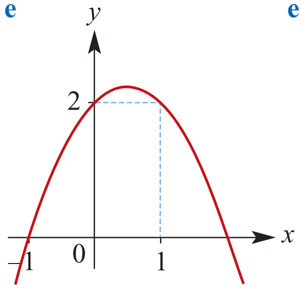
and $\quad k = 2$

$\therefore$ $\quad y = 2(x - 1)^2 + 6$

and the rule is $y = 2(x^2 - 2x + 1) + 6$

$y = 2x^2 - 4x + 8$

e



e This is of the form $y = ax^2 + bx + c$

For $(-1, 0)$ $\quad 0 = a - b + c \quad (1)$

For $(0, 2)$ $\quad 2 = c \quad (2)$

For $(1, 2)$ $\quad 2 = a + b + c \quad (3)$

Substitute $c = 2$ in (1) and (3)

$$0 = a - b + 2$$

$$-2 = a - b \quad (1a)$$

$$0 = a + b \quad (3a)$$

Substract (3a) from (1a)

$$-2 = -2b$$

Therefore $\quad b = 1$

Substitute $b = 1$ and $c = 2$ in (1)

Therefore $\quad 0 = a - 1 + 2$

$$0 = a + 1$$

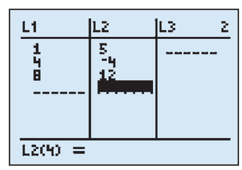
and hence $\quad a = -1$

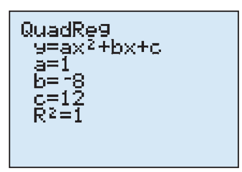
Thus the quadratic rule is $\quad y = -x^2 + x + 2$

Using a graphics calculator

When three arbitrary points are given, **5:QuadReg** from the **CALC** submenu of **STAT** can be used. For example, if it is known that a parabola passes through the points with coordinates (1, 5), (4, −4) and (8, 12), enter these as x- and y-values in lists 1 and 2 respectively. These are accessed by pressing $\boxed{\text{STAT}}$ and choosing **1:Edit**. Clear list L1 by taking the cursor over the top of L1 and pressing $\boxed{\text{CLEAR}}$ and $\boxed{\text{ENTER}}$.

In the Home screen enter **QuadReg L1, L2** and press $\boxed{\text{ENTER}}$. The result is as shown. The equation of the parabola is $y = x^2 - 8x + 12$.





Exercise 3L

Example 37a

1 A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with coordinates (2, 8). Find the value of a.

Example 37b

2 A quadratic rule for a particular parabola is of the form $y = ax^2 + c$. The parabola passes through the points with coordinates (−1, 4) and (0, 8). Find the value of a and of c.

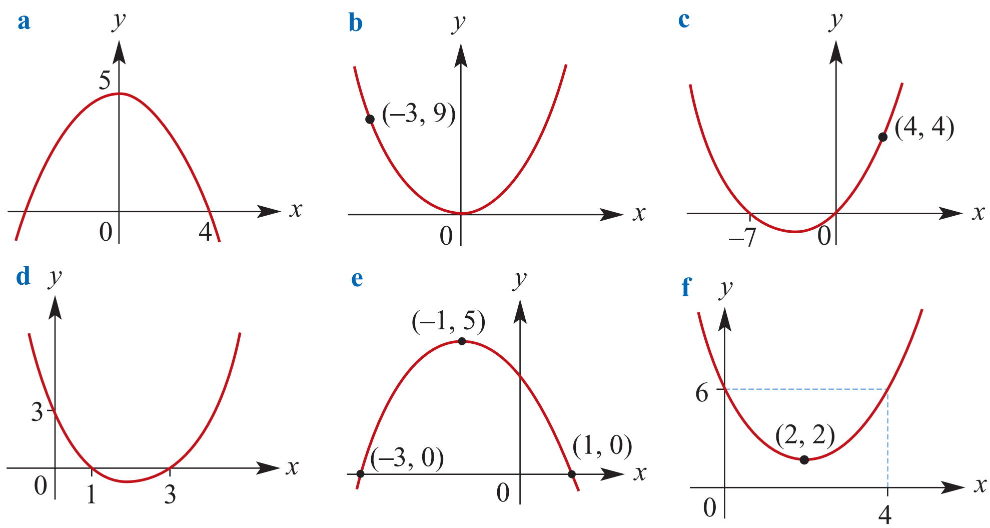
Example 37c

3 A quadratic rule for a particular parabola is of the form $y = ax^2 + bx$. The parabola passes through the points with coordinates $(-1, 4)$ and one of its x-axis intercepts is 6. Find the value of a and of b.

Example 37d

4 A quadratic rule for a particular parabola is of the form $y = a(x - b)^2 + c$. The parabola has vertex $(1, 6)$ and passes through the point with coordinates $(2, 4)$. Find the values of a, b and c.

5 Determine the equation of each of the following parabolas:

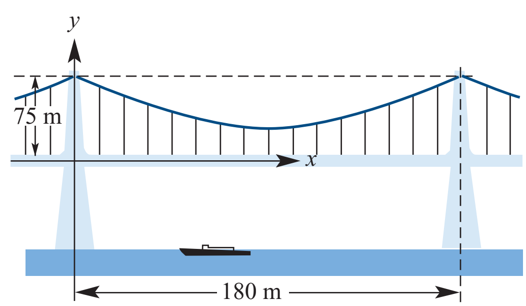


6 A parabola has vertex with coordinates $(-1, 3)$ and passes through the point with coordinates $(3, 8)$. Find the equation for the parabola.

7 A parabola has x-axis intercepts 6 and -3 and passes through the point $(1, 10)$. Find the equation of the parabola.

8 A parabola has vertex with coordinates $(-1, 3)$ and y-axis intercept 4. Find the equation for the parabola.

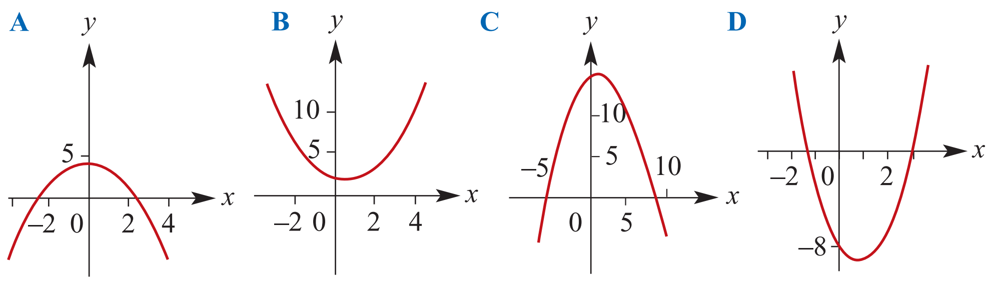
9 Assuming that the suspension cable shown in the diagram forms a parabola, find the rule which describes its shape. The minimum height of the cable above the roadway is 30 m.



10 Which of the curves could be most nearly defined by each of the following?

a $y = \frac{1}{3}(x + 4)(8 - x)$ **b** $y = x^2 - x + 2$

c $y = -10 + 2(x - 1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$

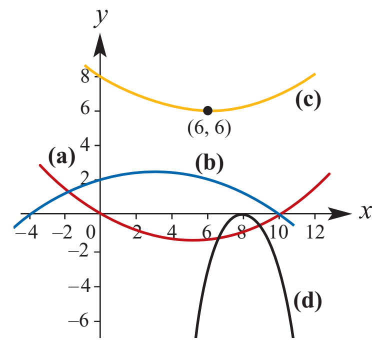


11 A parabola has the same shape as $y = 2x^2$ but its turning point is $(1, -2)$. Write its equation.

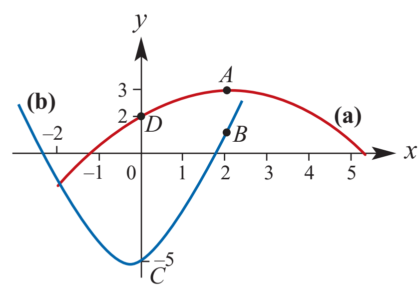
12 A parabola has its vertex at $(1, -2)$ and passes through the point $(3, 2)$. Write its equation.

13 A parabola has its vertex at $(2, 2)$ and passes through $(4, -6)$. Write its equation.

14 Write down four quadratic rules that have graphs similar to those in the diagram.



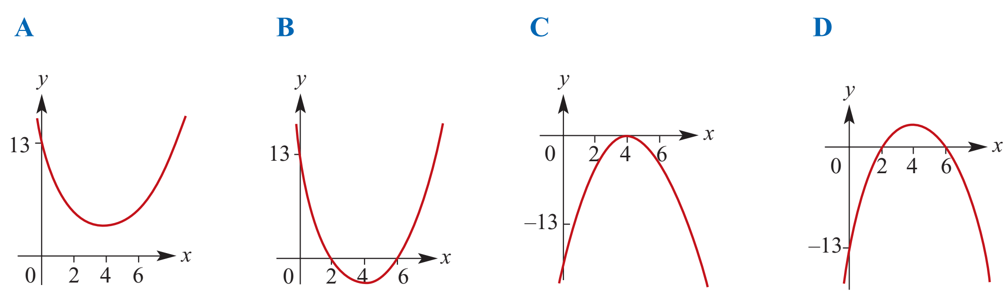
15 Find quadratic expressions which could represent the two curves in this diagram, given that the coefficient of x is 1 in each case. A is $(2, 3)$, B is $(2, 1)$, C is $(0, -5)$ and D is $(0, 2)$



16 The rate of rainfall during a storm t hours after it began was 3 mm per hour when $t = 5$, 6 mm per hour when $t = 9$ and 5 mm per hour when $t = 13$. Assuming that a quadratic model applies, find an expression for the rate, r mm per hour, in terms of t.

17 **a** Which of the graphs shown below could represent the graph of the equation $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?

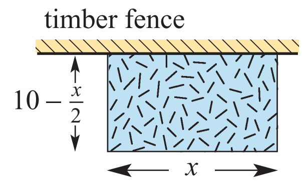


3.13 Quadratic models

Example 38

Jenny wishes to fence off a rectangular vegetable garden in her backyard. She has 20 m of fencing wire which she will use to fence three sides of the garden, with the existing timber fence forming the fourth side. Calculate the maximum area she can enclose.

Solution



Let A = area of the rectangular garden

x = length of the garden

$$\therefore \text{ width} = \frac{20 - x}{2} = 10 - \frac{x}{2}$$

$$\therefore \quad A = x\left(10 - \frac{x}{2}\right)$$

$$= 10x - \frac{x^2}{2}$$

$$= -\frac{1}{2}(x^2 - 20x + 100 - 100)$$

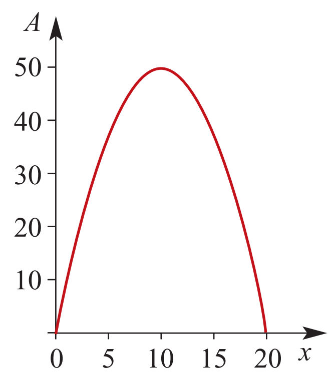
(completing the square)

$$= -\frac{1}{2}(x^2 - 20x + 100) + 50$$

$$= -\frac{1}{2}(x - 10)^2 + 50$$

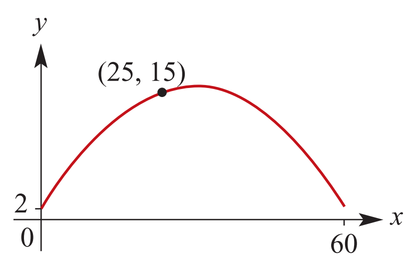
$\therefore$ the maximum area is 50 m^2 when $x = 10$.

The graph of the relation is shown.



Example 39

A cricket ball is thrown by a fielder. It leaves his hand at a height of 2 metres above the ground and the wicketkeeper takes the ball 60 metres away again at a height of 2 metres. It is known that after the ball has gone 25 metres it is 15 metres above the ground. The path of the cricket ball is a parabola with equation $y = ax^2 + bx + c$.



a Find the values of a, b, and c.

b Find the maximum height of the ball above the ground.

c Find the height of the ball 5 metres horizontally before it hits the wicketkeeper's gloves.

Solution

a The data can be used to obtain three equations:

$$\begin{aligned} 2 &= c & (1)\\ 15 &= (25)^2a + 25b + c & (2)\\ 2 &= (60)^2a + 60b + c & (3) \end{aligned}$$

Substitute equation (1) in equations (2) and (3).

$$\begin{aligned} \therefore 13 &= 625a + 25b & (1')\\ 0 &= 3600a + 60b & (2') \end{aligned}$$

Simplify (2′) by dividing both sides by 60.

$$0 = 60a + b \qquad (2')$$

Multiply this by 25 and subtract from equation (1′).

$$13 = -875a$$

$$\therefore a = -\frac{13}{875} \quad \text{and} \quad b = \frac{156}{175}$$

$$\therefore y = -\frac{13}{875}x^2 + \frac{156}{175}x + 2$$

b The maximum height occurs when $x = 30$ and $y = \frac{538}{35}$.

$\therefore$ maximum height is $\frac{538}{35}$ m.

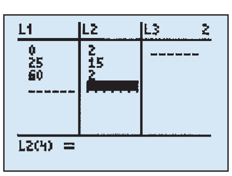
c When $x = 55$, $y = \frac{213}{35}$

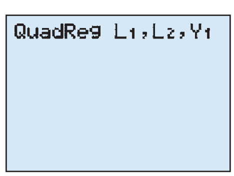
$\therefore$ height of ball is $\frac{213}{35}$ m.

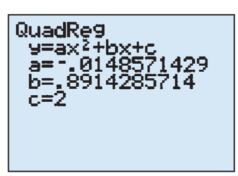
Using a graphics calculator

This equation can also be obtained on a graphics calculator. Enter the data in **L1** and **L2** as shown and then select **QuadReg** from the **CALC** submenu of **STAT**.

Complete in the Home screen as shown. Press ENTER to obtain the values of a, b, and c.







Obtain the graph of the quadratic by pressing GRAPH for a suitably chosen window.

Exercise 3M

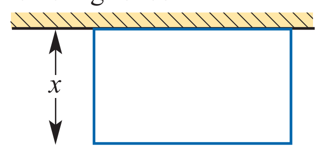
Example 38

1 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence.

a If the width of paddock is x m and the area inside the yard A m^2, write down the rule connecting A and x.

b Sketch the graph of A against x.

c Determine the maximum area that can be formed for the yard.



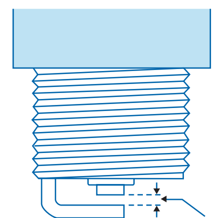
2 The efficiency rating, E, of a particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$.

a Sketch the graph of E against x for $0 \le x \le 1$.

b What values of x give a zero efficiency rating?

c What value of x gives the maximum efficiency rating?

d Use the graph, or otherwise, to determine the values of x between which the efficiency rating is 70 or more.



3 A piece of wire 68 cm in length is bent into the shape of a rectangle.

a If x cm is the length of the rectangle and A cm^2 is the area enclosed by the rectangular shape, write down a formula which connects A and x.

b Sketch the graph of A against x for suitable x-values.

c Use your graph to determine the maximum area formed.

4 A construction firm has won a contract to build cable-car pylons at various positions on the side of a mountain. Because of difficulties associated with construction in alpine areas, the construction firm will be paid an extra amount, C(\$), given by the formula $C = 240h + 100h^2$, where h is the height in km above sea level.

a Sketch the graph of C as a function of h. Comment on the possible values of h.

b Does C have a maximum value?

c What is the value of C for a pylon built at an altitude of 2500 m?

5 A tug-o-war team produces a tension in a rope described by the rule $T = 290(8t - 0.5t^2 - 1.4)$ units when t is the number of seconds after commencing the pull.

a Sketch a graph of T against t, stating the practical domain.

b What is the greatest tension produced during a ‘heave’?

6 A cricketer struck a cricket ball such that its height, d metres, after it had travelled x metres horizontally was given by the rule $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2$, $x \geq 0$.

a Use a graphics calculator to graph d against x for values of x ranging from 0 to 30.

b **i** What was the maximum height reached by the ball?

ii If a fielder caught the ball when it was 2 m above the ground, how far was the ball from where it was hit?

iii At what height was the ball when it was struck?

7 Find the equation of the quadratic which passes through the points with coordinates:

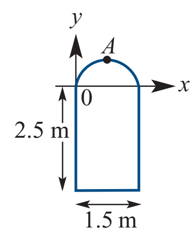
a $(-2, -1), (1, 2), (3, -16)$

b $(-1, -2), (1, -4), (3, 10)$

c $(-3, 5), (3, 20), (5, 57)$

Example 39

8 An arch on the top of a door is parabolic in shape. The point A is 3.1 m above the bottom of the door. The equation $y = ax^2 + bx + c$ can be used to describe the arch. Find the values of a, b, and c.



9 It is known that the daily spending of a government department follows a quadratic model. Let t be the number of days after 1 January and s be the spending in hundreds of thousands of dollars on a particular day, where $s = at^2 + bt + c$.

t	30	150	300
s	7.2	12.5	6

a Find the values of a, b and c.

b Sketch the graph for $0 \leq t \leq 360$. (Use a graphics calculator.)

c Find an estimate for the spending when:

i $t = 180$ **ii** $t = 350$

MC TEST D&D TEST

Chapter summary

- The general expression for a quadratic function is $y = ax^2 + bx + c$.
- **Expansions**

e.g. **i** $2x(x - 3) = 2x^2 - 6x$

ii $$\begin{aligned}(2x - 3)(3x + 4) &= 2x(3x + 4) - 3(3x + 4)\\ &= 6x^2 + 8x - 9x - 12\\ &= 6x^2 - x - 12\end{aligned}$$

iii $(x + a)^2 = x^2 + 2ax + a^2$

iv $(x - a)(x + a) = x^2 - a^2$

- **Factorising**

Type 1 Removing the highest common factor

e.g. $9x^3 + 27x^2 = 9x^2(x + 3)$

Type 2 Difference of two squares: $x^2 - a^2 = (x - a)(x + a)$

e.g. $16x^2 - 36 = (4x - 6)(4x + 6)$

Type 3 Grouping of terms

e.g. $$\begin{aligned}x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12)\\ &= x^2(x + 4) - 3(x + 4)\\ &= (x^2 - 3)(x + 4)\end{aligned}$$

Type 4 Factorising quadratic expressions

e.g. **i** $x^2 + 2x - 8 = (x + 4)(x - 2)$

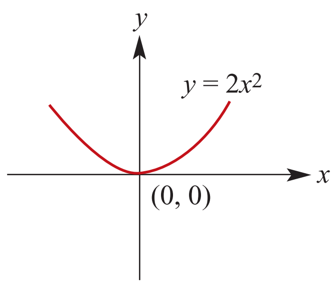
ii $6x^2 - 13x - 15$

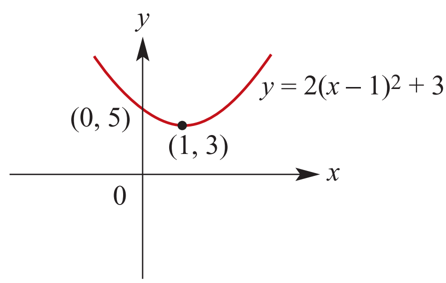
$6x^2$	-15	-13
$6x$	$+5$	$+5x$
x	-3	-18
		$-13x$

$6x^2 - 13x - 15 = (6x + 5)(x - 3)$

Review

- The graphs of a quadratic may be sketched by first expressing the rule in the form $y = a(x - h)^2 + k$. The graph of a quadratic of this form is obtained by translating the graph of $y = ax^2$ h units in the positive direction of the x-axis and k units in the positive direction of the y-axis (h, k positive).
  e.g. for $y = 2(x - 1)^2 + 3$





- A quadratic equation may be solved by:

  i Factorising

  e.g. $2x^2 + 5x - 12 = 0$

  $(2x - 3)(x + 4) = 0 \quad \therefore x = \dfrac{3}{2}$ or -4

  ii Completing the square

  e.g. $x^2 + 2x - 4 = 0$

  Add and subtract $\left(\dfrac{b}{2}\right)^2$ to 'complete the square'

  $$x^2 + 2x + 1 - 1 - 4 = 0$$
  $$\therefore\ (x + 1)^2 - 5 = 0$$
  $$\therefore\ (x + 1)^2 = 5$$
  $$\therefore\ x + 1 = \pm\sqrt{5} \quad \therefore\ x = -1 \pm \sqrt{5}$$

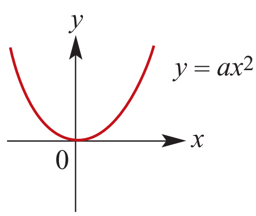
  iii Using the general quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

  e.g. $-3x^2 - 12x - 7 = 0$

  $$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$$
  $$= \frac{6 \pm \sqrt{15}}{-3}$$

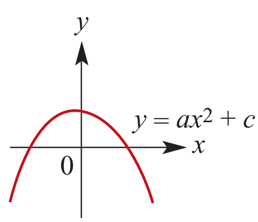
- The following can be used for sketching the graph of the quadratic relation $y = ax^2 + bx + c$:
  - **i** If $a > 0$, the function has a minimum value.
  - **ii** If $a < 0$, the function has a maximum value.
  - **iii** The value of c gives the y-axis intercept.
  - **iv** The equation of the axis of symmetry is $x = -\dfrac{b}{2a}$.
  - **v** The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$.
- From the general solution for the quadratic equation $ax^2 + bx + c = 0$, the number of x-axis intercepts can be determined:
  - **i** If $b^2 - 4ac > 0$, the equation has two distinct real roots a and b.
  - **ii** If $b^2 - 4ac = 0$, there is one root, $-\dfrac{b}{2a}$.
  - **iii** If $b^2 - 4ac < 0$, the equation has no real roots.
- To find a quadratic rule to fit given points:

i



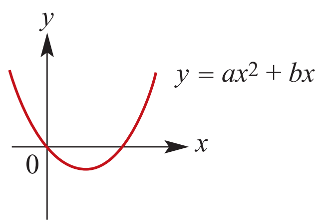
a can be calculated if one point is known.

ii



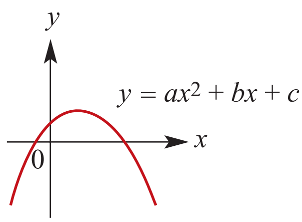
Two points are needed to determine a and c.

iii



Two points required to determine a and b.

iv



Three points required to determine a, b and c.

Multiple-choice questions

1 The linear factors of $12x^2 + 7x - 12$ are

A $4x - 3$ and $3x + 4$ **B** $3x - 4$ and $4x + 3$ **C** $3x - 2$ and $4x + 6$
D $3x + 2$ and $4x - 6$ **E** $6x + 4$ and $2x - 3$

2 The solutions of the equation $x^2 - 5x - 14 = 0$ are

A $x = -7$ only **B** $x = -7, x = 2$ **C** $x = -2, x = 7$ **D** $x = -2, x = -7$
E $x = -2$ only

3 For $y = 8 + 2x - x^2$, the maximum value of y is

A $-3\frac{1}{4}$ **B** $5\frac{1}{4}$ **C** 9 **D** $9\frac{1}{2}$ **E** 10

4 If the graph of $y = 2x^2 - kx + 3$ touches the x-axis then the possible values of k are

A $k = 2$ or $k = -3$ **B** $k = 1$ **C** $k = -3$ or $k = -\frac{1}{2}$

D $k = 1$ or $k = 3$ **E** $k = 2\sqrt{6}$ or $k = -2\sqrt{6}$

5 The solutions of the equation $x^2 - 56 = x$ are

A $x = -8$ or 7 **B** $x = -7$ or 8 **C** $x = 7$ or 8 **D** $x = -9$ or 6

E $x = 9$ or -6

6 The value of the discriminant of $x^2 + 3x - 10$ is

A 5 **B** -5 **C** 49 **D** 7 **E** -2

7 The coordinates of the turning point of the graph with equation $y = 3x^2 + 6x - 1$ are

A $\left(\frac{1}{3}, -2\right)$ **B** $\left(-\frac{1}{3}, 2\right)$ **C** $\left(-\frac{1}{3}, -4\right)$ **D** $(1, -4)$ **E** $(-1, -4)$

8 The equation $5x^2 - 10x - 2$ in turning point form $a(x - h)^2 + k$, by completing the square, is

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$

D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

9 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two real roots is (are)

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

10 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$

D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

Short-answer questions (technology-free)

1 Express each of the following in the form $(ax + b)^2$:

a $x^2 + 9x + \frac{81}{4}$ **b** $x^2 + 18x + 81$ **c** $x^2 - \frac{4}{5}x + \frac{4}{25}$

d $x^2 + 2bx + b^2$ **e** $9x^2 - 6x + 1$ **f** $25x^2 + 20x + 4$

2 Expand each of the following products:

a $-3(x - 2)$ **b** $-a(x - a)$

c $(7a - b)(7a + b)$ **d** $(x + 3)(x - 4)$

e $(2x + 3)(x - 4)$ **f** $(x + y)(x - y)$

g $(a - b)(a^2 + ab + b^2)$ **h** $(2x + 2y)(3x + y)$

i $(3a + 1)(a - 2)$ **j** $(x + y)^2 - (x - y)^2$

k $u(v + 2) + 2v(1 - u)$ **l** $(3x + 2)(x - 4) + (4 - x)(6x - 1)$

3 Express each of the following as a product of factors:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$

d $4 - x^2$ **e** $au + 2av + 3aw$ **f** $4a^2b^2 - 9a^4$

g $1 - 36x^2a^2$ **h** $x^2 + x - 12$ **i** $x^2 + x - 2$

j $2x^2 + 3x - 2$ **k** $6x^2 + 7x + 2$ **l** $3x^2 - 8x - 3$

m $3x^2 + x - 2$ **n** $6a^2 - a - 2$ **o** $6x^2 - 7x + 2$

4 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = -2x^2 + 3$ **c** $y = 2(x - 2)^2 + 3$

d $y = 2(x + 2)^2 + 3$ **e** $y = 2(x - 4)^2 - 3$ **f** $y = 9 - 4x^2$

g $y = 3(x - 2)^2$ **h** $y = 2(2 - x)^2 + 3$

5 Express in the form $y = a(x - h)^2 + k$ and hence sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 6x$ **c** $y = x^2 - 8x + 4$

d $y = 2x^2 + 8x - 4$ **e** $y = -3x^2 - 12x + 9$ **f** $y = -x^2 + 4x + 5$

6 Find:

i the x- and y-intercepts **ii** the axis of the symmetry **iii** the turning point

and hence sketch the graphs of each of the following:

a $y = x^2 - 7x + 6$ **b** $y = -x^2 - x + 12$ **c** $y = -x^2 + 5x + 14$

d $y = x^2 - 10x + 16$ **e** $y = 2x^2 + x - 15$ **f** $y = 6x^2 - 13x - 5$

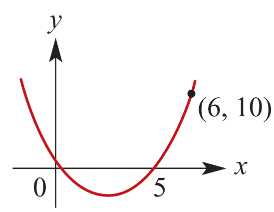
g $y = 9x^2 - 16$ **h** $y = 4x^2 - 25$

7 Use the quadratic formula to solve each of the following:

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 9x + 12 = 0$ **c** $x^2 - 4x + 2 = 0$

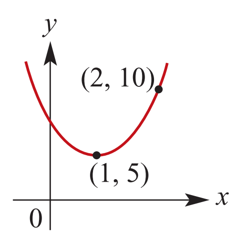
d $2x^2 + 7x + 2 = 0$ **e** $2x^2 + 7x + 4 = 0$ **f** $3x^2 + 9x - 1 = 0$

8 Find the equation of the quadratic, the graph of which is shown.



9 A parabola has the same shape as $y = 3x^2$ but its vertex is at (5, 2). Find the equation corresponding to this parabola.

10 Find the rule of the quadratic relation which describes the graph.



11 Find the coordinates of the point of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$ **b** $y = 8x + 11$ and $y = 2x^2$

c $y = 3x^2 + 7x$ and $y = 2$ **d** $y = 2x^2$ and $y = 2 - 3x$

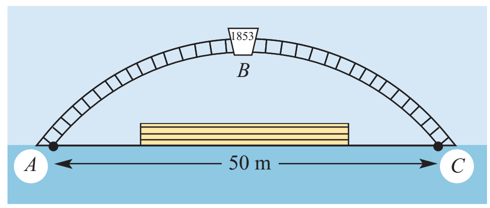
12 **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution.

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions.

c Show that there are real solutions of the equation $4mx^2 + 4(m - 1)x + m - 2 = 0$ for all real x.

Extended-response questions

1 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. AC is the water level.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

2 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other a rectangular shape in which the length is twice its width.

a If x is the side length of the square, write down the dimensions of the rectangle in terms of x.

b Formulate a rule for A, the combined area of the square and rectangle in cm^2, in terms of x.

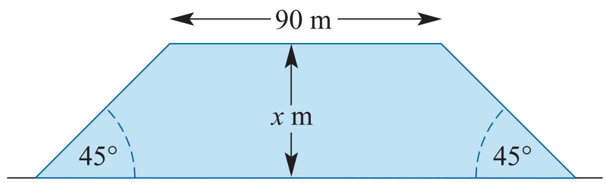
c Determine the lengths of the two parts if the sum of the areas is to be a minimum.

3 Water is pumped into an empty metal tank at a steady rate of 0.2 litres/min. After 1 hour the depth of water in the tank is 5 cm; after 5 hours the depth is 10 cm.

a If the volume of water in the tank is V litres when the depth is x cm and there is a quadratic relationship between V and x, write down a rule which connects V and x.

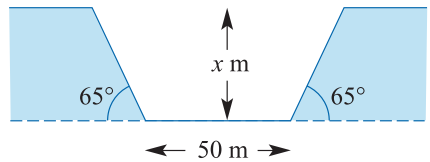
b It is known that the maximum possible depth of water in the tank is 20 cm. For how long, from the beginning, can water be pumped into the tank at the same rate without overflowing?

4 The figure shows a section view of a freeway embankment to be built across a flood-prone river flat. The height of the embankment is x m and width at the top is 90 m.



a Find a formula, in terms of x, for V, the volume of earth in m^3 required to build a 120 m length of freeway embankment.

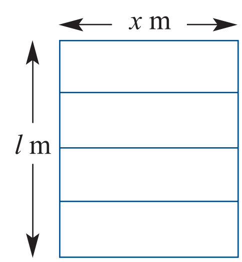
This figure shows another section of the freeway which is to be constructed by cutting through a hillside. The depth of the cutting is x m and the width of the cutting at the base is 50 m.



b Find a formula for the volume of earth, in m^3, which would have to be excavated to form a straight 100 m section of the cutting.

c If $x = 4$ m, what length of embankment could be constructed from earth taken from the cutting?

5 100 m of angle steel is used to make a rectangular frame with three crossbars as shown in the figure.



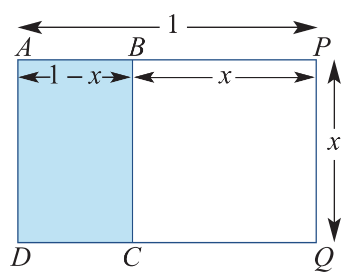
a If the width of the frame is x m, determine an expression for l, the length of the frame in metres, in terms of x.

b The frame is to be covered by light aluminium sheeting. If the area of this sheeting is A m^2, formulate a rule connecting A and x.

c Sketch a graph of A against x, stating the axes intercepts and the turning point.

d What is the maximum area and the value of x which gives this area?

6 A shape which has been of interest to architects and artists over the centuries is the 'golden rectangle'. Many have thought that it gave the perfect proportions for buildings. The rectangle is such that if a square is drawn on one of the longer sides then the new rectangle is similar to the original.

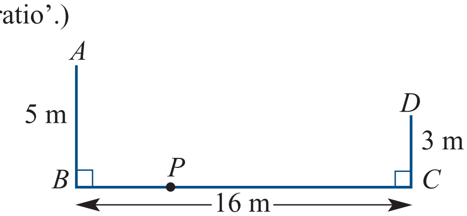


Let the length of $AP = 1$ unit, then

$AB = 1 - x$ units and $\dfrac{AP}{AD} = \dfrac{AD}{AB}$.

Find the value of x. (x is known as the 'golden ratio'.)

7 P is a point on line BC x m from B.



a Find distance PA in terms of x.

b **i** Find distance PC in terms of x.

ii Find distance PD in terms of x.

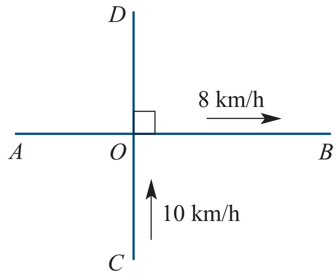
c Find x if $PA = PD$.

d Find x if $PA = 2PD$. (Answer correct to 3 decimal places.)

e Find x if $PA = 3PD$. (Answer correct to 3 decimal places.)

Review

8 AB and CD are crossroads. A jogger runs along road AB at a speed of 8 km/hour and passes O at 1.00 pm. Another runner is moving along road CD. The second runner is moving at 10 km/hour and passes O at 1.30 pm.



a Let y km be their distance apart t hours after 1.00 pm.

 i Find an expression for y in terms of t.

 ii Plot the graph of y against t on a graphics calculator.

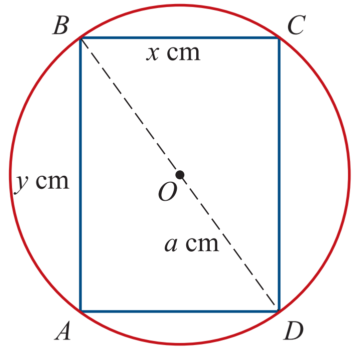
 iii Find the time(s) when the runners are 4 km apart. (Use a graphics calculator.)

 iv Find the time at which the runners are closest and their distance apart at this time.

b Find the exact value(s) of t for which:

 i $y = 5$ **ii** $y = 6$

9 A rectangle of perimeter b cm is inscribed in a circle of radius a cm. The rectangle has width x cm and length y cm.



a Apply Pythagoras' theorem in triangle BCD to show that $x^2 + y^2 = 4a^2$.

b Form a second equation involving x, y and b.

c Eliminate y from these equations to form a quadratic equation in terms of x, a and b.

d As x, y and $2a$ are the sides of a triangle, $x + y > 2a$. Use this result and apply the discriminant to the quadratic equation formed in part **c** to show that the rectangle can be inscribed in the circle only if $4a < b \le 4\sqrt{2}a$.

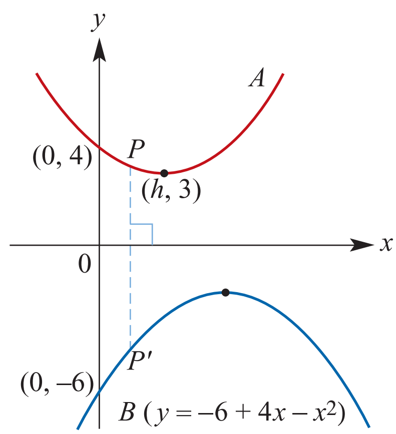
e **i** If $a = 5$ and $b = 24$, find the value(s) of x and y.

 ii If $b = 4\sqrt{2}a$, find the value of x and y in terms of a.

f If $\dfrac{b}{a} = 5$, find the values of x and y in terms of a.

g Write a program to solve the quadratic equation found in part c for suitable choices of a and b and state the values of x and y. (Answers correct to 2 decimal places.)

10 The equation of curve B is $y = -6 + 4x - x^2$.



a $(h, 3)$ is the vertex of a parabola A, with equation $y = x^2 + bx + c$. Find the values of b, c and h for $h > 0$.

b Let P be a point on curve A, and P' be a point on curve B such that PP' is perpendicular to the x-axis.

 i The coordinates of P are $(x, x^2 + bx + c)$. State the coordinates of P' in terms of x.

 ii Find the coordinates of M, the midpoint of PP' in terms of x.

 iii Find the coordinates of M for $x = 0, 1, 2, 3, 4$.

 iv Give the equation of the straight line on which all of these points lie. (This is called the locus of the midpoints.)

c Let d be the distance PP'.

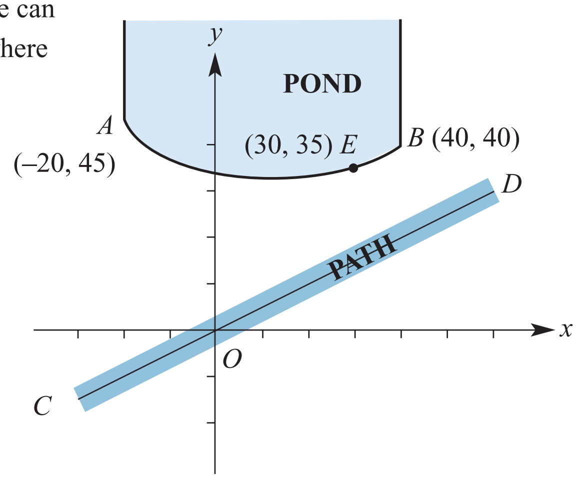
i Express d in terms of x.

ii Sketch the graph of d against x.

iii Find the minimum value of d and the value of x for which this occurs.

11 A path cuts across a park. Its centreline can be described by the equation $y = \frac{x}{2}$, where the origin is at a point O in the park.

The path starts at a point $C(-30, -15)$ and finishes at a point $D(60, 30)$.



a How long is the path?

One boundary of the pond in the park is parabolic in shape. The boundary passes through the points $A(-20, 45)$, $B(40, 40)$ and $E(30, 35)$. The equation of the parabola is of the form $y = ax^2 + bx + c$.

Graphics CALCULATOR

b **i** Find the equation of the parabola.

ii Find the coordinates of the vertex of the parabola.

c On the one set of axes sketch the graphs of $y = \frac{x}{2}$ and the parabola. (Use a graphics calculator to help.)

d Consider the rule $y = (ax^2 + bx + c) - \frac{1}{2}x$, where a, b and c have been determined in part b i.

i What does this expression determine?

ii Find the minimum value of this expression and the value of x for which this occurs.