CHAPTER

5

Functions, Relations and Transformations

Objectives

- To understand and use the **notation of sets**, including the symbols $\in$, $\subseteq$, $\cap$, $\cup$, $\emptyset$ and $\backslash$.
- To use the notation for **sets of numbers**.
- To understand the concept of **relation**.
- To understand the terms **domain** and **range**.
- To understand the concept of **function**.
- To understand the term **one-to-one**.
- To understand the terms **implied (maximal) domain, restriction of a function** and **hybrid function**.
- To be able to find the inverse of a one-to-one function.
- To define dilations from the axes, reflections in the axes and translations.
- To be able to apply transformations to graphs of relations.
- To apply a knowledge of functions to **solving problems**.

Sections 5.1 and 5.2 of this chapter introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

5.1 Set notation and sets of numbers

Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

By a **set** we mean a collection of objects. The objects that are in the set are known as **elements** or members of the set. If x is an element of a set A we write $x \in A$. This can also be read as 'x is a member of the set of A' or 'x belongs to A' or 'x is in A'.

The notation $x \notin A$ means x is *not* an element of A. For example:

$$2 \notin \text{set of odd numbers}$$

Set B is called a subset of set A, *if* and *only if*

$$x \in B \text{ implies } x \in A$$

To indicate that B is a subset of A, we write $B \subseteq A$.

This expression can also be read as 'B is contained in A' or 'A contains B'.

The set of elements common to two sets A and B is called the **intersection** of A and B and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.

The **union** of sets A and B, written $A \cup B$, is the set of elements that are either in A or in B. This does not exclude objects that are elements of both A and B.

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution

a $A \cap B = \{3, 7\}$ **b** $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

In Example 1, $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

Finally we introduce the **set difference** of two sets A and B:

$$A \setminus B = \{x : x \in A, x \notin B\}$$

$A \setminus B$ is the set of elements of A that are not elements of B. For sets A and B in Example 1, $A \setminus B = \{1, 2\}$ and $B \setminus A = \{4, 5, 6\}$.

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \ldots\}$ are called the **natural numbers** and the elements of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are called **integers**.

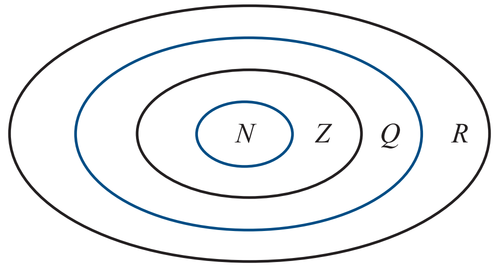
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rationals are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised by the property that each rational number may be written as a terminating or recurring decimal.

The set of real numbers will be denoted by R.

The set of rational numbers will be denoted by Q.



The set of integers will be denoted by Z.

The set of natural numbers will be denoted by N.

It is clear that $N \subseteq Z \subseteq Q \subseteq R$, and this may be represented by this diagram.

The set of all x such that $(\ldots)$ is denoted by $\{x: (\ldots)\}$.

Thus $\{x: 0 < x < 1\}$ is the set of all real numbers between 0 and 1.

$\{x: x > 0, x \text{ rational}\}$ is the set of all positive rational numbers.
$\{2n: n = 0, 1, 2, \ldots\}$ is the set of all even numbers.

Among the most important subsets of R are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers and that $a < b$:

$$(a, b) = \{x: a < x < b\} \qquad [a, b] = \{x: a \le x \le b\}$$
$$(a, b] = \{x: a < x \le b\} \qquad [a, b) = \{x: a \le x < b\}$$
$$(a, \infty) = \{x: a < x\} \qquad [a, \infty) = \{x: a \le x\}$$
$$(-\infty, b) = \{x: x < b\} \qquad (-\infty, b] = \{x: x \le b\}$$

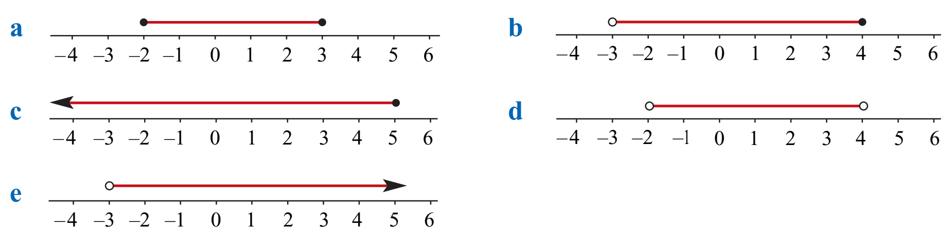
Intervals may be represented by diagrams as shown in Example 2.

Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution



The 'closed' circle (●) indicates that the number is included.
The 'open' circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notations:

$R^+ = \{x: x > 0\}$

$R^- = \{x: x < 0\}$

$R \setminus \{0\}$ is the set of real numbers excluding 0.

$Z^+ = \{x: x \in Z, x > 0\}$

Exercise 5A

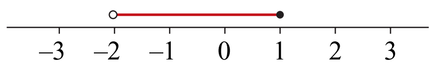
Example 1

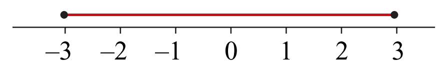
1 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$, $C = \{1, 7, 11, 25, 30\}$, find:

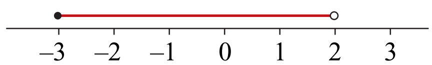
a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup B$

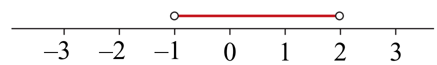
d $A \setminus B$ **e** $C \setminus B$ **f** $A \cap C$

2 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc:

a 

b 

c 

d 

Example 2

3 Illustrate each of the following intervals on a number line:

a $[-3, 4)$ **b** $(-\infty, 3]$ **c** $[-2, -1]$

d $(-2, \infty)$ **e** $(-2, 3)$ **f** $(-2, 4]$

4 Use the appropriate interval notation, i.e. $[a, b]$, (a, b) etc., to describe each of the following sets:

a $\{x: -1 \le x \le 2\}$ **b** $\{x: -4 < x \le 2\}$ **c** $\{y: 0 < y < \sqrt{2}\}$

d $\left\{y: -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}}\right\}$ **e** $\{x: x > -1\}$ **f** $\{x: x \le -2\}$

g R **h** $R^+ \cup \{0\}$ **i** $R^- \cup \{0\}$

5 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$, $C = \{1, 7, 11, 25, 30\}$, find:

a $[-3, 8] \cap C$ **b** $(-2, 10] \cap B$ **c** $(3, \infty) \cap B$ **d** $(2, \infty) \cup B$

6 For each of the following use one number line on which to represent the sets:

a $[-2, 5], [3, 4], [-2, 5] \cap [3, 4]$ **b** $[-2, 5], R \setminus [-2, 5]$

c $[3, \infty), (-\infty, 7], [3, \infty) \cap (-\infty, 7]$ **d** $[-2, 3], R \setminus [-2, 3]$

5.2 Relations, domain and range

In previous chapters we have looked at how to sketch the graphs of various mathematical relations. We will now look at this aspect of defining relations in a more formal way.

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first element and y the second.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$ **b** $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets. The set of all the first elements of the ordered pairs is called the **domain**. The set of all the second elements of ordered pairs is called the **range**.

In the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{(x, y): y = x + 1, x \in \{1, 2, 3, 4\}\}$$

is the relation

$$\{(1, 2)(2, 3)(3, 4)(4, 5)\}$$

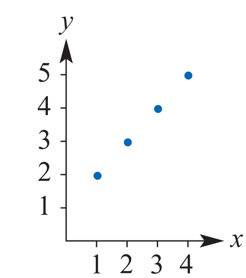
The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

Graphing relations

We can represent a relation as a graph on a set of cartesian axes.

On the right is the graph of the relation

$$\{(x, y): y = x + 1, x \in \{1, 2, 3, 4\}\}$$

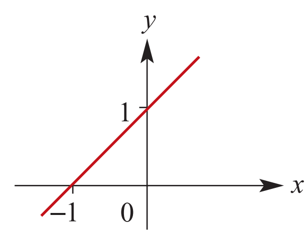


Note that we only graph the individual points of this relation.

If the domain of the relation is the set of real numbers, R, then there is an infinite number of points and the graph of

$$y = x + 1, x \in R$$

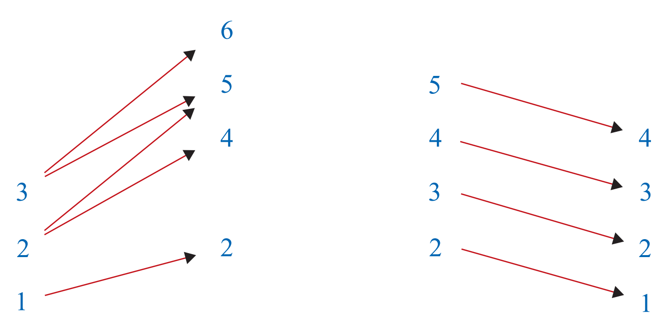
is a continuous straight line.



Mapping diagrams

A relation may be represented by a mapping diagram.

Mapping diagram 1 *Mapping diagram 2*



Mapping diagram 1 represents the relation $\{(3, 6), (3, 5), (2, 5), (2, 4), (1, 2)\}$.
Mapping diagram 2 represents the relation $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$.

In summary, a relation may be written as:

i a listed set of ordered pairs (not always convenient or possible)

ii a rule with a specified or implied domain.

A relation may be represented by a graph or a mapping diagram.

Example 3

Sketch a graph of each of the following and state its domain and range:

a $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

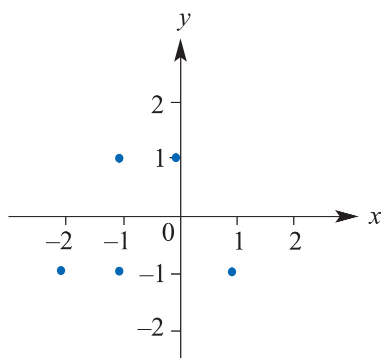
b $\{(x, y): x^2 + y^2 = 1, x \in [-1, 1]\}$

c $\{(x, y): 2x + 3y = 6, x \geq 0\}$

d $\{(x, y): y = 2x - 1, x \in [-1, 2]\}$

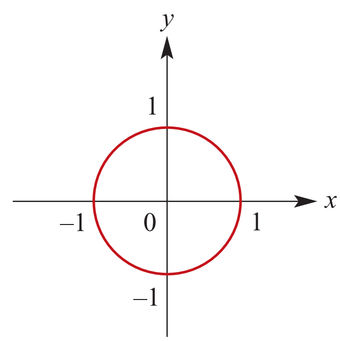
Solution

a



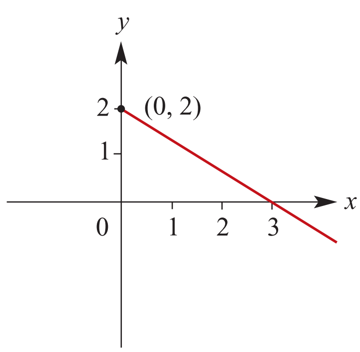
Domain $= \{-2, -1, 0, 1\}$
Range $= \{-1, 1\}$

b



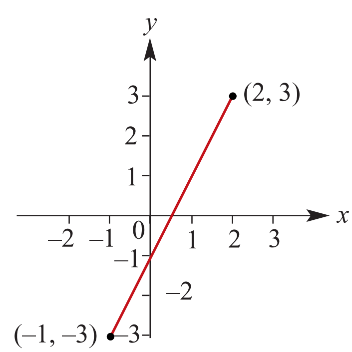
Domain $= \{x: -1 \leq x \leq 1\}$
Range $= \{y: -1 \leq y \leq 1\}$

c



Domain $= R^+ \cup \{0\}$
Range $= (-\infty, 2]$

d



Domain $= [-1, 2]$
Range $= [-3, 3]$

Often set notation is not used in the specification of a relation.

For example:

- $\{(x, y): y = x^2\}$ is written as $y = x^2$
- $\{(x, y): y = x + 1\}$ is written as $y = x + 1$.

This has been the case in your previous considerations of relations.

Note: In order to determine the range of a relation it is necessary to consider the graph. This strategy is used in the following examples.

Example 4

Find the range of the relation with rules:

a $y = x^2 - 4x + 5$ **b** $y = -x^2 + 4x - 5$

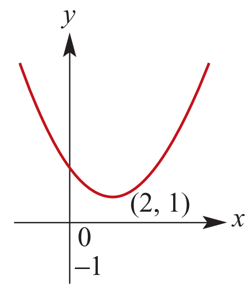
Solution

a Completing the square gives

$$y = x^2 - 4x + 5 = (x - 2)^2 + 1$$

The vertex of the corresponding parabola is at the point with coordinates (2, 1).

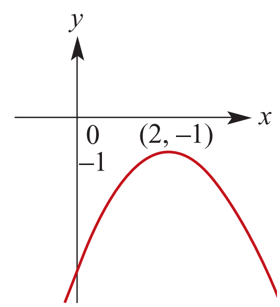
Therefore the minimum value of the relation is 1 and the range is $[1, \infty)$.



b Completing the square gives

$$y = -x^2 + 4x - 5$$
$$= -(x^2 - 4x + 5) = -(x - 2)^2 - 1$$

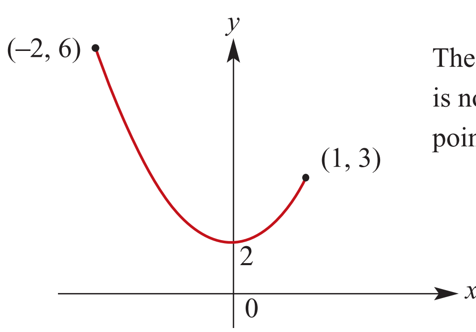
The vertex of the corresponding parabola is at the point with coordinates $(2, -1)$.
Therefore the maximum value of the relation is -1 and the range is $(-\infty, -1]$.



Example 5

Sketch the graph of the relation $y = x^2 + 2$ for $x \in [-2, 1]$ and state the range.

Solution



The range is [2, 6]. Note that the range is not determined by considering the end points alone.

Implied (maximal) domain

When the rule for a relation is written and no domain is stipulated then it is understood that the domain taken is the largest for which the rule has meaning. This domain is called the **maximal or implied domain**.

For example, the maximal domain of $y = x^2$ is R and for $x^2 + y^2 = 1$ the maximal domain is $[-1, 1]$. This concept is considered again in Section 5.3.

Example 6

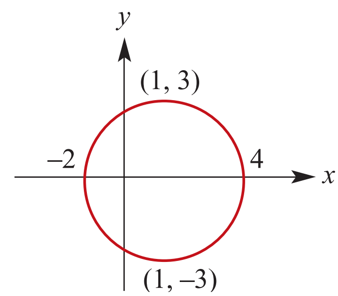
For each of the following relations state the maximal domain and the range:

a $(x-1)^2 + y^2 = 9$ **b** $y = \sqrt{9-x^2} + 1$

Solution

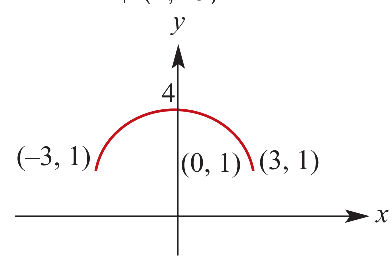
a This relation is a circle with centre (1, 0) and radius 3 units.

The maximal domain is $[-2, 4]$ and the range is $[-3, 3]$.



b This relation is a semicircle with centre (0, 1) and radius 3 units.

The maximal domain is $[-3, 3]$ and the range is $[1, 4]$.

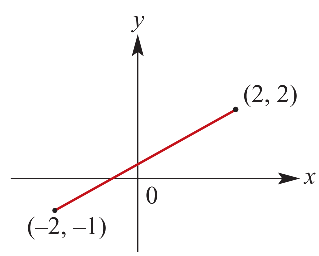


Exercise 5B

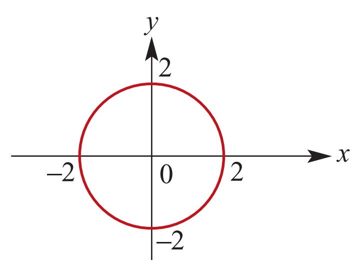
Example 3

1 State the domain and range for the relations represented by each of the following graphs:

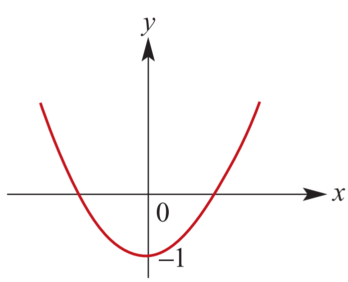
a



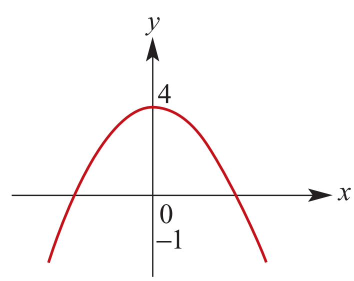
b



c



d



Example 3 **2** Sketch the graphs of each of the following and state the range of each:

a $y = x + 1 \quad x \in [2, \infty)$ **b** $y = -x + 1 \quad x \in [2, \infty)$

c $y = 2x + 1 \quad x \in [-4, \infty)$ **d** $y = 3x + 2 \quad x \in (-\infty, 3)$

e $y = x + 1 \quad x \in (-\infty, 3]$ **f** $y = -3x - 1 \quad x \in [-2, 6]$

g $y = -3x - 1 \quad x \in [-5, -1]$ **h** $y = 5x - 1 \quad x \in (-2, 4)$

Examples 4, 5 **3** Sketch the graphs of each of the following relations, stating the range of each:

a $\{(x, y): y = x^2 + 1\}$ **b** $\{(x, y): y = x^2 + 2x + 1\}$

c $\{(x, y): y = 4 - x^2, x \in [-2, 2]\}$ **d** $\{(x, y): y = x^2 + 2x + 3\}$

e $\{(x, y): y = -x^2 + 2x + 3\}$ **f** $\{(x, y): y = x^2 - 2, x \in [-1, 2]\}$

g $\{(x, y): y = 2x^2 - 3x + 6\}$ **h** $\{(x, y): y = 6 - 3x + x^2\}$

Example 6 **4** Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $\{(x, y): x^2 + y^2 = 9\}$ **b** $(x - 2)^2 + (y - 3)^2 = 16$

c $(2x - 1)^2 + (2y - 4)^2 = 1$ **d** $y = \sqrt{25 - x^2}$

e $y = -\sqrt{25 - x^2}$ **f** $\left\{(x, y): y = -\sqrt{25 - (x - 2)^2}\right\}$

5.3 Functions

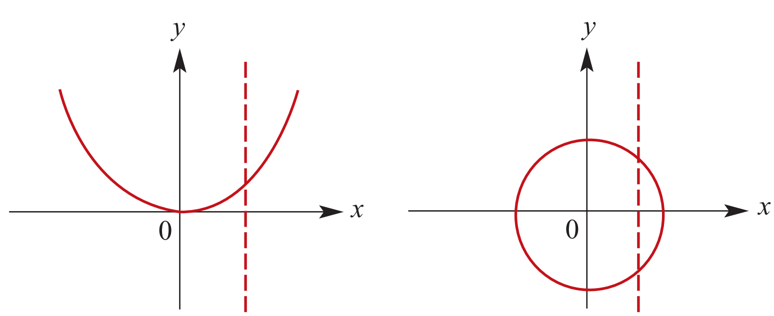
A **function** is a relation for which for each x-value of an ordered pair there is a unique y-value of the ordered pair. This means that if (a, b) and (a, c) are ordered pairs of a function then $b = c$.

One way to identify if a relation is a function is to draw a graph of the relation and apply the **vertical line test**.

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, the relation is a **function**.

Examples:

$y = x^2$ is a function $\qquad$ $x^2 + y^2 = 4$ is not a function



Function notation

Functions are usually denoted with lower case letters such as f, g, h.

The definition of a function tells us that for each x in the domain there is a unique element y in the range such that $(x, y) \in f$.

The element y is called the **image** of x under f or the **value** of f at x and x is called the **pre-image** of y.

Since the y-value obtained is a **function** of the x-value that was substituted into the rule, we use the notation $f(x)$ (read 'f of x') in place of y.

i.e. Instead of $y = 2x + 1$ we write $f(x) = 2x + 1$.
$f(2)$ means the y-value obtained when $x = 2$.

e.g.
$$f(2) = 2(2) + 1 = 5$$
$$f(-4) = 2(-4) + 1 = -7$$
$$f(a) = 2a + 1$$

By incorporating the mapping notation we have an alternative way of writing functions.

i For the function $\{(x, y): y = x^2\}$, with domain $= R$ we write $f: R \to R, f(x) = x^2$.

ii For the function $\{(x, y): y = 2x - 1, x \in [0, 4]\}$ we write $f: [0, 4] \to R, f(x) = 2x - 1$.

iii For the function $\{(x, y): y = \frac{1}{x}\}$ with domain $= R \setminus \{0\}$ we write $f: R \setminus \{0\} \to R, f(x) = \frac{1}{x}$.

If the domain is R we often just write the rule. For example in **i** $f(x) = x^2$.

Note that in using the notation $f: X \to Y$, X is the domain but Y is not necessarily the range. It is a set that contains the range and is called the **co-domain**. With this notation for function we write domain of f as dom f and range of f as ran f.

A function $f: R \to R, f(x) = a$ is called a **constant function**.

For such a f, dom $f = R$ and ran $f = \{a\}$, e.g. let $f(x) = 7$. The dom $f = R$ and ran $f = \{7\}$.

A function $f: R \to R, f(x) = mx + c$ is called a **linear function**, e.g. let $f(x) = 3x + 1$.

Then dom $f = R$ and ran $f = R$.

Note if the domain of a linear function is R and $m \neq 0$, the range of $f = R$.

Example 7

If $f(x) = 2x^2 + x$, find $f(3)$, $f(-2)$ and $f(x - 1)$.

Solution

$$\begin{aligned} f(3) &= 2(3)^2 + 3 = 21 \\ f(-2) &= 2(-2)^2 - 2 = 6 \\ f(x-1) &= 2(x-1)^2 + x - 1 \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

Example 8

If $f(x) = 2x + 1$, find $f(-2)$ and $f\left(\frac{1}{a}\right)$, $a \neq 0$.

Solution

$$f(-2) = 2(-2) + 1 = -3$$
$$f\left(\frac{1}{a}\right) = 2\left(\frac{1}{a}\right) + 1 = \frac{2}{a} + 1$$

Example 9

Consider the function defined by $f(x) = 2x - 4$ for all $x \in R$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

Solution

a
$$\begin{aligned} f(2) &= 2(2) - 4 \\ &= 0 \\ f(-1) &= 2(-1) - 4 \\ &= -6 \\ f(t) &= 2t - 4 \end{aligned}$$

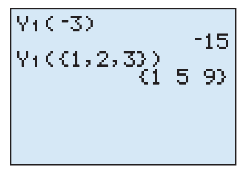
b
$$\begin{aligned} f(t) &= t \\ \therefore 2t - 4 &= t \\ \therefore t - 4 &= 0 \\ \therefore t &= 4 \end{aligned}$$

c
$$\begin{aligned} f(x) &\geq x \\ \therefore 2x - 4 &\geq x \\ \therefore x - 4 &\geq 0 \\ \therefore x &\geq 4 \end{aligned}$$

Using a graphics calculator

The graphics calculator uses function notation with the evaluation of functions.

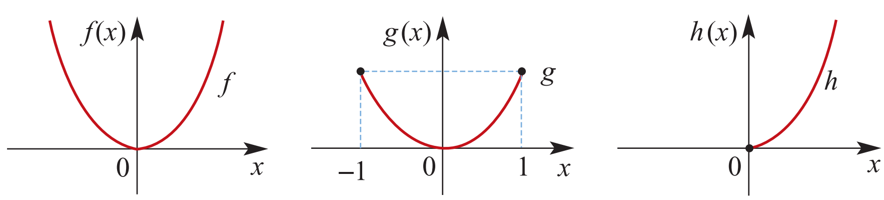
For example for $f(x) = 4x - 3$ enter Y1 $= 4x - 3$ in the **Y=** screen. In the Home screen enter Y1 (-3) to evaluate $f(-3)$ and enter Y1($\{1, 2, 3\}$) to evaluate a list of values.



Note: The symbol Y1 is found by pressing VARS, and selecting **Y-VARS** and then **1:Function** and finally **1:Y$_1$**.

Restriction of a function

Consider the following functions:



$f(x) = x^2, x \in R$ $\quad$ $g(x) = x^2, -1 \leq x \leq 1$ $\quad$ $h(x) = x^2, x \in R^+ \cup \{0\}$

The different letters, f, g, and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains. We call g and h restrictions of f, since their domains are subsets of the domain of f.

Example 10

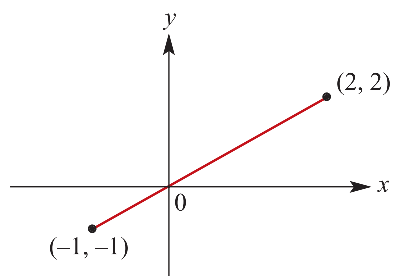
Sketch the graph of each of the following functions and state its range.

a $f: [-1, 2] \to R, f(x) = x$ **b** $f: [-1, 1] \to R, f(x) = x^2 + x$

c $f: (0, 2] \to R, f(x) = \dfrac{1}{x}$ **d** $f: R \to R, f(x) = x^2 - 2x + 8$

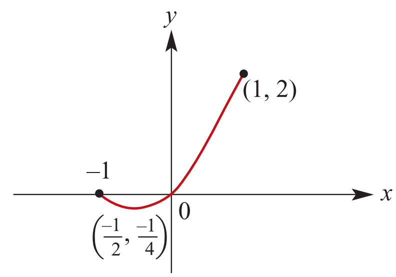
Solution

a



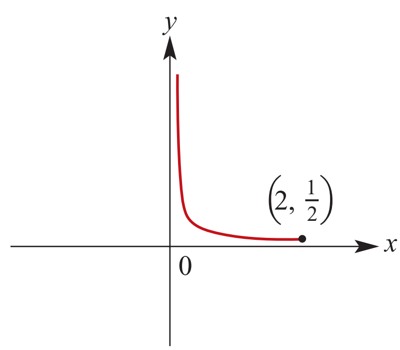
Range is $[-1, 2]$

b



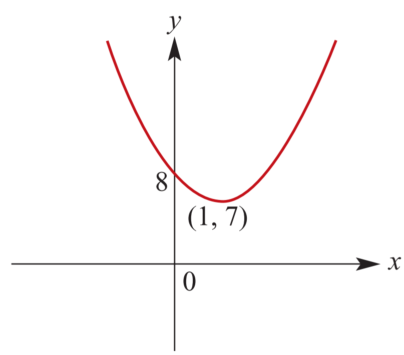
Range is $\left[\dfrac{-1}{4}, 2\right]$

c



Range is $\left[\dfrac{1}{2}, \infty\right)$

d



$f(x) = x^2 - 2x + 8 = (x - 1)^2 + 7$

Range is $[7, \infty)$

Exercise 5C

1 Sketch the graph of each of the following relations, then state the range of each and specify whether the relation is a function or not:

a $y = x^2, x \in [0, 4]$ **b** $\{(x, y): x^2 + y^2 = 4\}, x \in [0, 2]$

c $\{(x, y): 2x + 8y = 16, x \in [0, \infty)\}$ **d** $y = \sqrt{x}, x \in R^+$

e $\{(x, y): y = \dfrac{1}{x^2}, x \in R \setminus \{0\}\}$ **f** $\{(x, y): y = \dfrac{1}{x}, x \in R^+\}$

g $y = x^2, x \in [-1, 4]$ **h** $\{(x, y): x = y^2, x \in R^+\}$

2 Which of the following relation are functions? State the domain and range for each:

a $\{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4)\}$

b $\{(-2, -1), (-1, -2), (0, 2), (1, 4), (2, -5)\}$

c $\{(0, 1), (0, 2), (-1, 2), (3, 4), (5, 6)\}$
d $\{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$
e $\{(x, -2): x \in R\}$
f $\{(3, y): y \in Z\}$
g $y = -x + 3$
h $y = x^2 + 5$
i $\{(x, y): x^2 + y^2 = 9\}$

Example 7

3 **a** Given that $f(x) = 2x - 3$, find:
i $f(0)$ **ii** $f(4)$ **iii** $f(-1)$ **iv** $f(6)$

b Given that $g(x) = \dfrac{4}{x}$, find:
i $g(1)$ **ii** $g(-1)$ **iii** $g(3)$ **iv** $g(2)$

c Given that $g(x) = (x - 2)^2$, find:
i $g(4)$ **ii** $g(-4)$ **iii** $g(8)$ **iv** $g(a)$

Example 8

d Given that $f(x) = 1 - \dfrac{1}{x}$, find:
i $f(1)$ **ii** $f(1 + a)$ **iii** $f(1 - a)$ **iv** $f\left(\dfrac{1}{a}\right)$

4 Find the value(s) of x for which the function has the given value:
a $f(x) = 5x - 2, f(x) = 3$
b $f(x) = \dfrac{1}{x}, f(x) = 6$
c $f(x) = x^2, f(x) = 9$
d $f(x) = (x + 1)(x - 4), f(x) = 0$
e $f(x) = x^2 - 2x, f(x) = 3$
f $f(x) = x^2 - x - 6, f(x) = 0$

5 Let $g(x) = x^2 + 2x$ and $h(x) = 2x^3 - x^2 + 6$.
a Evaluate $g(-1)$, $g(2)$ and $g(-2)$.
b Evaluate $h(-1)$, $h(2)$ and $h(-2)$.
c Express the following in terms of x:
i $g(-3x)$ **ii** $g(x - 5)$ **iii** $h(-2x)$
iv $g(x + 2)$ **v** $h(x^2)$

6 Consider the function $f(x) = 2x^2 - 3$. Find:
a $f(2), f(-4)$ **b** the range of f

Example 9

7 Consider the function $f(x) = 3x + 1$. Find:
a the image of 2 **b** the pre-image of 7 **c** $\{x: f(x) = 2x\}$

8 Consider the function $f(x) = 3x^2 + 2$. Find:
a the image of 0 **b** the pre-image(s) of 5 **c** $\{x: f(x) = 11\}$

9 Consider the functions $f(x) = 7x + 6$ and $g(x) = 2x + 1$. Find:
a $\{x: f(x) = g(x)\}$ **b** $\{x: f(x) > g(x)\}$ **c** $\{x: f(x) = 0\}$

10 Rewrite each of the following using the $f: X \to Y$ notation:
a $\{(x, y): y = 3x + 2\}$
b $\{(x, y): 2y + 3x = 12\}$
c $\{(x, y): y = 2x + 3, x \geq 0\}$
d $y = 5x + 6, -1 \leq x \leq 2$
e $y + x^2 = 25, -5 \leq x \leq 5$
f $y = 5x - 7, 0 \leq x \leq 1$

Example 10

11 Sketch the graphs of each of the following functions and state the range of each:

a $f: [-1, 2] \to R,\ f(x) = x^2$ **b** $f: [-2, 2] \to R,\ f(x) = x^2 + 2x$

c $f: (0, 3] \to R,\ f(x) = \dfrac{1}{x}$ **d** $f: R \to R,\ f(x) = x^2 - 2x + 3$

5.4 Special types of functions and implied domains

Types of relations

There are four types of relations:

1 One-to-one Each x-value maps onto a unique y-value. e.g. $y = 2x$

2 Many-to-one More than one x-value maps onto the same y-value. e.g. $y = x^2$

3 One-to-many An x-value maps onto more than one y-value. e.g. $y = \pm\sqrt{x}$

4 Many-to-many More than one x-value maps onto more than one y-value. e.g. $x^2 + y^2 = 4$

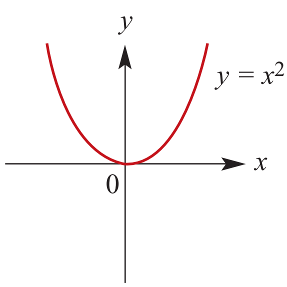
As stated earlier, a function is a relation in which no two ordered pairs have the same x-value. Functions are relations that are either **many-to-one** or **one-to-one**.

One-to-one functions

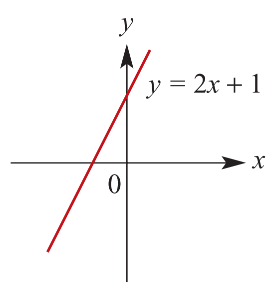
The **vertical line test** can be used to determine whether a relation is a function or not.

Similarly there is a geometric test that determines whether a function is **one-to-one** or not.

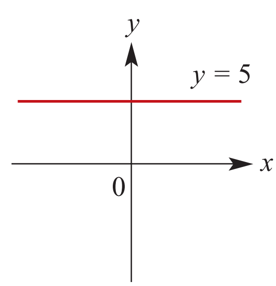
If a horizontal line can be drawn anywhere on the graph of a function and only ever intersects the graph a maximum of once, the function is **one-to-one**.



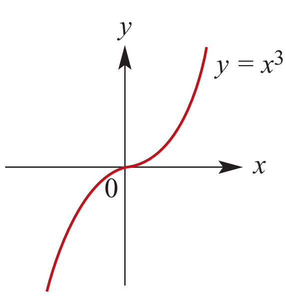
not one-to-one



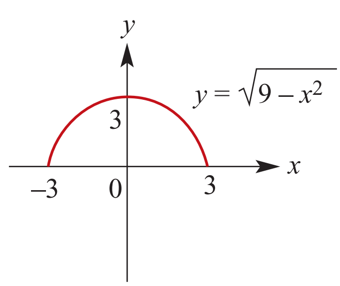
one-to-one



not one-to-one



one-to-one



not one-to-one

When the domain of a relation is not explicitly stated, it is assumed to consist of all real numbers for which the rule has a meaning. We refer to the **implied (maximal)** domain of a relation, because the domain is implied by the rule.

For example:

$S = \{(x, y): y = x^2\}$ has the **implied (maximal) domain** R.

and $T = \{(x, y): y = \sqrt{x}\}$ has the **implied (maximal) domain** $[0, \infty)$.

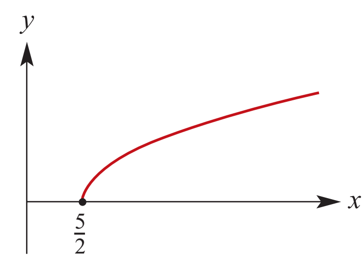
Example 11

State the maximal domain, sketch the graph and find the corresponding range of each of the following:

a $y = \sqrt{2x - 5}$ **b** $y = \dfrac{1}{2x - 5}$

Solution

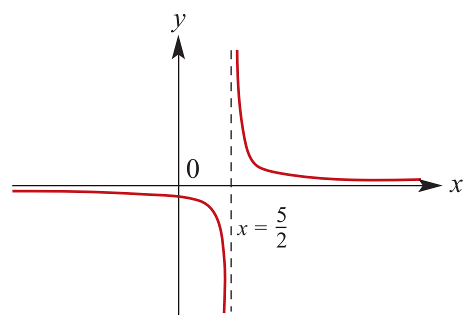
a To be defined $2x - 5 \geq 0$. That is, $x \geq \frac{5}{2}$. Hence the maximal domain is $[\frac{5}{2}, \infty)$.



The range of the function is $R^+ \cup \{0\} = [0, \infty)$.

The range of the function is $R \setminus \{0\}$.

b To be defined $2x - 5 \neq 0$, i.e. $x \neq \frac{5}{2}$. Hence maximal domain is $R \setminus \{\frac{5}{2}\}$ and the range is $R \setminus \{0\}$.



Exercise 5D

1 State which of the following functions are one-to-one:

a $\{(1, 3), (2, 4), (4, 4), (3, 6)\}$ **b** $\{(1, 3), (2, 4), (3, 6), (7, 9)\}$

c $\{(x, y): y = x^2\}$ **d** $\{(x, y): y = 3x + 1\}$

e $f(x) = x^3 + 1$ **f** $f(x) = 1 - x^2$

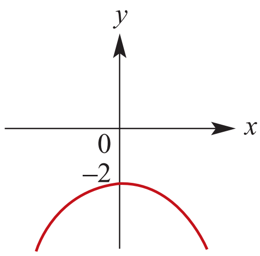
g $y = x^2, x \geq 0$

2 The following are the graphs of a relation:

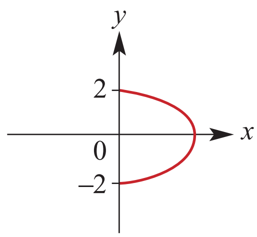
i State which are the graphs of a function.

ii State which are the graphs of a one-to-one function.

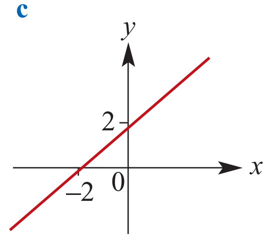
a



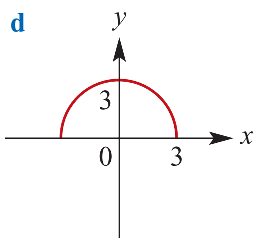
b



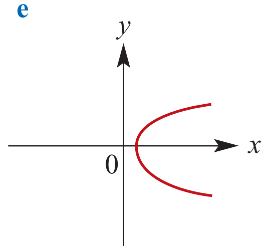
c



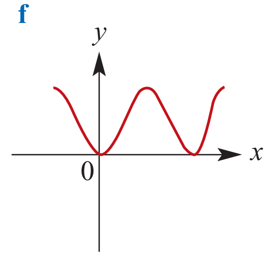
d



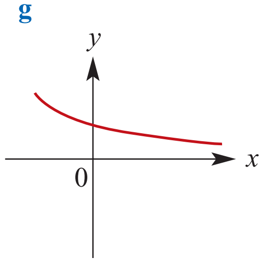
e



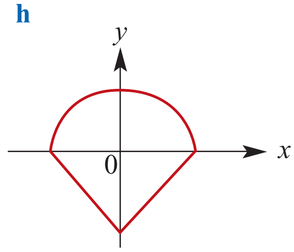
f



g



h



Example 11

3 For each of the following, find the maximal domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$ **b** $y = 2\sqrt{x}$ **c** $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$ **e** $y = \dfrac{1}{\sqrt{x}}$ **f** $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$ **h** $y = \sqrt{2x - 1}$ **i** $y = \sqrt{3 - 2x}$

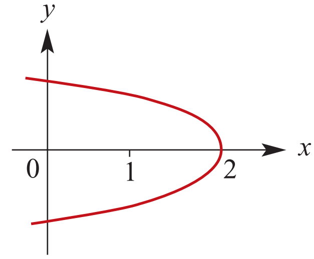
j $y = \dfrac{1}{2x - 1}$ **k** $y = \dfrac{1}{(2x - 1)^2} - 3$ **l** $y = \dfrac{1}{2x - 1} + 2$

4 Each of the following is the rule of a function. In each case write down the maximal domain and the range:

a $f(x) = 3x + 4$ **b** $g(x) = x^2 + 2$

c $y = -\sqrt{16 - x^2}$ **d** $y = \dfrac{1}{x + 2}$

5 The graph shown is of the relation $\{(x, y): y^2 = -x + 2, x \leq 2\}$.
From this relation, form two functions and specify the range of each.



6 a Draw the graph of $f: R \to R, f(x) = x^2 - 2$.
b By restricting the domain of f, form two one-to-one functions that have the same rule as f.

5.5 Hybrid functions

Functions which have different rules for different subsets of the domain, are called **hybrid functions**.

Example 12

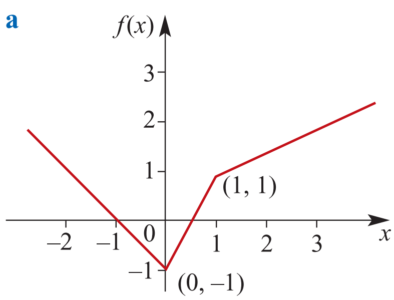
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \leq x \leq 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x \geq 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

Exercise 5E

Example 12
1 Sketch the graph of each of the following functions and state its range:

a $h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

b $h(x) = \begin{cases} x - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$

c $h(x) = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}$

d $h(x) = \begin{cases} 1 + x, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$

e $h(x) = \begin{cases} x, & x \geq 1 \\ 2 - x, & x < 1 \end{cases}$

2 **a** Sketch the graph of the function:

$$f(x) = \begin{cases} \frac{2}{3}x + 3, & x < 0 \\ x + 3, & 0 \leq x \leq 1 \\ -2x + 6, & x > 1 \end{cases}$$

b What is the range of f?

3 Sketch the graph of the function:

$$g(x) = \begin{cases} -x - 3, & x < 1 \\ x - 5, & 1 \leq x \leq 5 \\ 3x - 15, & x > 5 \end{cases}$$

4 **a** Sketch the graph of the function:

$$h(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$$

b State the range of h.

5 **a** Sketch the graph of the function:

$$f(x) = \begin{cases} x + 3, & x < -3 \\ x^2 - 9, & -3 \leq x \leq 3 \\ x - 3, & x > 3 \end{cases}$$

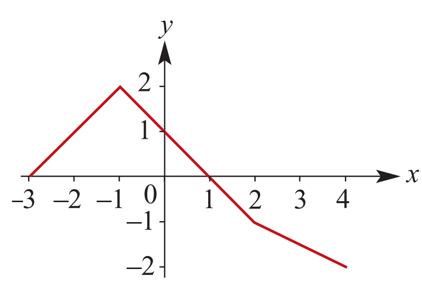
b State the range of f.

6 **a** Sketch the graph of the function:

$$f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ x, & x \leq 1 \end{cases}$$

b State the range of f.

7 Specify the function represented by this graph:



5.6 Miscellaneous exercises

Example 13

The volume of a sphere of radius r is determined by the function with rule $V(r) = \frac{4}{3}\pi r^3$. State the practical domain of the function V and find $V(10)$.

Solution

The practical domain is $(0, \infty)$.

$$\begin{aligned} V(10) &= \frac{4}{3} \times \pi \times 10^3 \\ &= \frac{4000\pi}{3} \end{aligned}$$

The volume is $\frac{4000\pi}{3}$ cubic units.

Example 14

If $f: R \to R, f(x) = ax + b, f(1) = 7$ and $f(5) = 19$, find a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$:

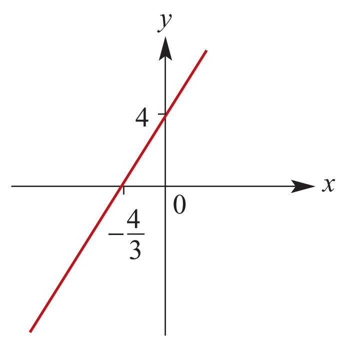
$$7 = a + b \qquad (1)$$

and $$19 = 5a + b \qquad (2)$$

Subtract (1) from (2):

$$\begin{aligned} 12 &= 4a \\ a &= 3, \text{ and substituting in (1) gives } b = 4 \\ a &= 3, b = 4 \end{aligned}$$

and $f(x) = 3x + 4$



Example 15

Find the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

4 and -2 are solutions to the quadratic equations $f(x) = 0$.

Thus $f(x) = k(x - 4)(x + 2)$

As $f(0) = 16$

$16 = k(-4)(2)$

$\therefore$ $k = -2$

and $$\begin{aligned} f(x) &= -2(x - 4)(x + 2) \\ &= -2(x^2 - 2x - 8) \\ &= -2x^2 + 4x + 16 \end{aligned}$$

Exercise 5F

Example 14

1 If $f(x) = a + bx$ and $f(4) = -1$ and $f(8) = 1$:

a Find a and b. **b** Solve the equation $f(x) = 0$.

2 Find a linear function f such that $f(0) = 7$, whose graph is parallel to that of the function with rule $g(x) = 2 - 5x$.

3 f is a linear function such that $f(-5) = -12$ and $f(7) = 6$.

a Find:

i $f(0)$ **ii** $f(1)$

b Solve the equation $f(x) = 0$.

4 If $f(x) = 2x + 5$, find:

a $f(p)$ **b** $f(p + h)$

c $f(p + h) - f(p)$ **d** $f(p + 1) - f(p)$

5 If $f(x) = 3 - 2x$, find $f(p + 1) - f(p)$.

6 A metal bar is L cm long when its temperature is C degrees centigrade. L and C are approximately related by the formula $L = 0.002C + 25$.

a L is a function of C and the rule can be written $L(C) = 0.002C + 25$. State a possible practical domain for the function.

b Find:

i $L(30)$ **ii** $L(16)$ **iii** $L(100)$ **iv** $L(500)$

Example 15

7 Find the quadratic function f such that $f(2) = f(4) = 0$ and 7 is the greatest value of $f(x)$.

8 Write $f(x) = x^2 - 6x + 16$ in the form $f(x) = (x - 3)^2 + p$ and hence state the range of f.

9 State the range of each of the following:

a $f(x) = -2x^2 + x - 2$ **b** $f(x) = 2x^2 - x + 4$

c $f(x) = -x^2 + 6x + 11$ **d** $g(x) = -2x^2 + 8x - 5$

10 $f: [-1, 6] \to R, f(x) = 5 - 3x$

a Sketch the graph of f. **b** State the range of f.

11 $f: [-1, 8] \to R, f(x) = (x - 2)^2$

a Sketch the graph of f. **b** State the range of f.

12 State the implied domain and range of each of the following relations:

a $x^2 + y^2 = 9$ **b** $(x - 2)^2 + y^2 = 1$

c $(2x - 1)^2 + (2y - 1)^2 = 1$ **d** $(x - 4)^2 + y^2 = 25$

e $(y - 2)^2 + x^2 = 16$

13 The domain of the function f is $\{1, 2, 3, 4\}$. Find the range of f if:

a $f(x) = 2x$ **b** $f(x) = 5 - x$ **c** $f(x) = x^2 - 4$ **d** $f(x) = \sqrt{x}$

14 $f: R \to R, f(x) = ax^2 + bx + c$. Find a, b and c if $f(0) = 2, f(4) = 0$ and $f(5) = 0$.

15 Find two quadratic functions f and g such that $f(1) = 0$, $g(1) = 0$ and $f(0) = 10$, $g(0) = 10$ and both have a maximum value of 18.

16 **a** Find the set of values of k for which $f(x) = 3x^2 - 5x - k$ is greater than 1 for all real x.

b Show that, for all k, the minimum value of $f(x)$ occurs when $x = \dfrac{5}{6}$. Find k if this minimum value is zero.

5.7 Inverse functions

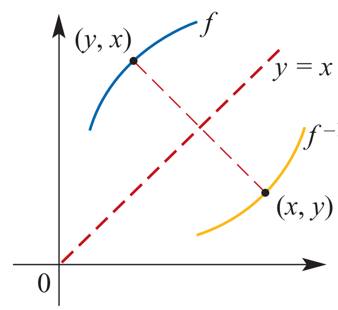
Excel

If f is a one-to-one function then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus, if f is a one-to-one function, a new function f^{-1}, called the inverse of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f$$

It is not difficult to see what the relation between f and f^{-1} means geometrically. The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f. Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.



From this the following is evident:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$

A function has an inverse function if and only if it is one-to-one.

CAS APPENDIX

Example 16

Find the inverse function f^{-1} of the function $f(x) = 2x - 3$ and sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes.

Solution

The graph of f has equation $y = 2x - 3$ and the graph of f^{-1} has equation $x = 2y - 3$, i.e. x and y are interchanged.

Solve for y:

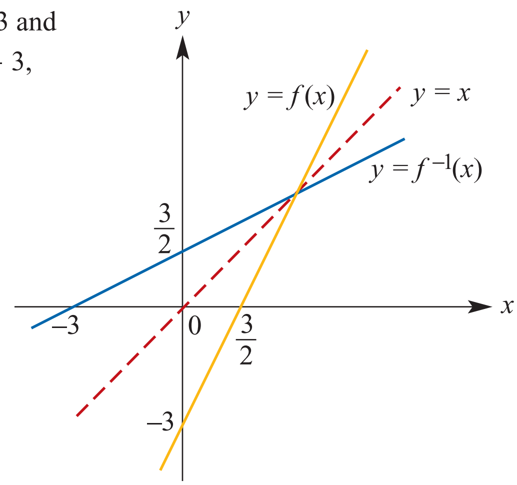
$$x + 3 = 2y$$

and $\quad y = \frac{1}{2}(x + 3)$

$\therefore \quad f^{-1}(x) = \frac{1}{2}(x + 3)$

$\text{dom } f = \text{ran } f^{-1} = R$

and $\quad \text{ran } f = \text{dom } f^{-1} = R$



Example 17

Let $f: [3, 6] \rightarrow R, f(x) = \left(\frac{x}{3}\right)^2$. Find f^{-1} and state its domain and range.

Solution

Let $y = \left(\frac{x}{3}\right)^2$. Then the inverse function has rule

$$x = \left(\frac{y}{3}\right)^2$$

$\therefore \quad \pm\sqrt{x} = \frac{y}{3}$

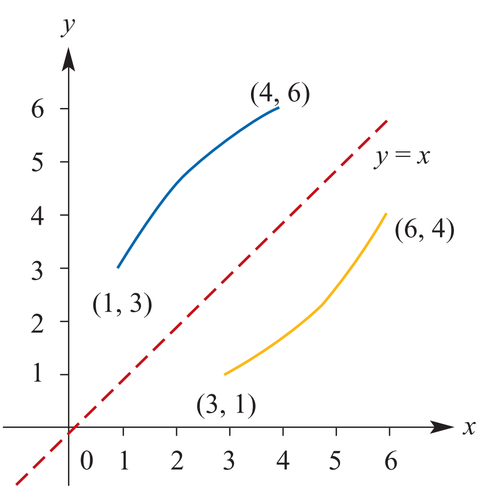
and $\quad y = \pm 3\sqrt{x}$

But ran $f^{-1} = \text{dom } f = [3, 6]$

$\therefore \quad f^{-1}(x) = 3\sqrt{x}$

and dom $f^{-1} = \text{ran } f = [1, 4]$

i.e. $f^{-1}: [1, 4] \rightarrow R, f^{-1}(x) = 3\sqrt{x}$



Example 18

Find the inverse of each of the following functions:

a $\{(1, 2), (3, 4), (5, 6), (7, 8)\}$

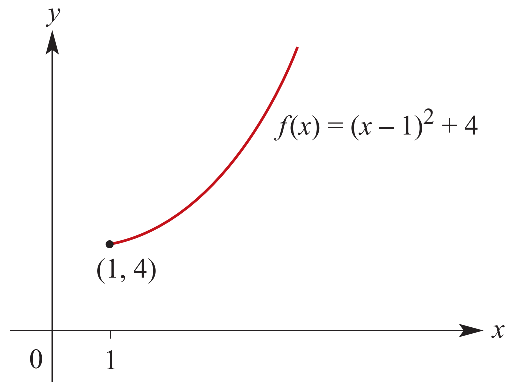
b $f: [1, \infty) \rightarrow R, f(x) = (x - 1)^2 + 4$

Solution

a $\{(2, 1), (4, 3), (6, 5), (8, 7)\}$

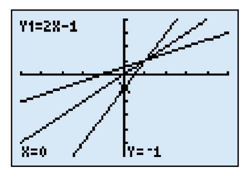
b The inverse has rule

$$
\begin{aligned}
x &= (y-1)^2 + 4 \\
\therefore \quad (y-1)^2 &= x - 4 \\
\therefore \quad y - 1 &= \pm\sqrt{x-4} \\
\therefore \quad y &= 1 \pm \sqrt{x-4} \\
\text{But ran } f^{-1} &= \text{dom } f = [1, \infty) \\
\therefore \quad f^{-1}(x) &= 1 + \sqrt{x-4} \\
\text{Also dom } f^{-1} &= \text{ran } f = [4, \infty)
\end{aligned}
$$



Using a graphics calculator

Drawing the graph of the inverse of a function



Enter Y1 = $2x - 1$ and Y2 = x in the **Y=** window and choose a suitable window. Go to the Home screen and enter **8:DrawInv** Y1 and press ENTER. The result is as shown.

Note: **8:DrawInv** is obtained from the **DRAW** menu (2ND PRGM) and the symbol Y1 is found by pressing VARS and selecting **Y-VARS**, then **1:Function** and finally **1:Y1**.

The graphs of $y = 2x - 1$, $y = x$ and the inverse of $y = 2x - 1$, which has rule $y = \dfrac{x+1}{2}$, are displayed.

Exercise 5G

Example 16

1 Find the inverse function of each of the following, clearly stating the domain and range of f^{-1}:

a $\{(1, 3), (-2, 6), (4, 5), (7, 1)\}$

b $f: R \to R,\ f(x) = 6 - 2x$

c $f: [1, 5] \to R,\ f(x) = 3 - x$

d $f: R^+ \to R,\ f(x) = x + 4$

Example 17

e $f: (-\infty, 4] \to R,\ f(x) = x + 4$

f $f: [0, \infty) \to R,\ f(x) = x^2$

Example 18

g $f: [2, \infty) \to R,\ f(x) = (x-2)^2 + 3$

h $f: (-\infty, 4] \to R,\ f(x) = (x-4)^2 + 6$

i $f: [0, 1] \to R,\ f(x) = \sqrt{1-x}$

j $f: [0, 4] \to R,\ f(x) = \sqrt{16 - x^2}$

k $f: [-1, 7] \to R,\ f(x) = 16 - 2x$

l $f: [0, \infty) \to R,\ f(x) = (x+4)^2 + 6$

2 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f(x) = 2x - 6$.

b Find the coordinates of the point for which $f(x) = f^{-1}(x)$.

3 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f : [0, \infty) \to R,\ f(x) = x^2$.

b Find the coordinates of the point(s) for which $f(x) = f^{-1}(x)$.

4 $f : R \to R,\ f(x) = ax + b$, where a and b are non-zero constants, and $f(1) = 2$ and $f^{-1}(1) = 3$. Find the values of a and b.

5 $f : (-\infty, a] \to R, f(x) = \sqrt{a - x}$.

a Find $f^{-1}(x)$.

b If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at $x = 1$, find the possible values for a.

5.8 Translations of functions[1]

In this and the following two sections a formal study of transformations is undertaken using function notation. These sections also provide systematic methods for applying and determining transformations.

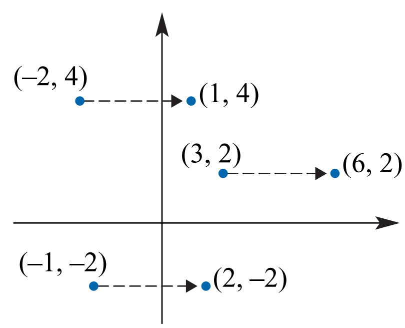
Let $R^2 = \{(x, y): x, y \in R\}$. That is R^2 is the set of all ordered pairs of real numbers. Transformations associate each ordered pair of R^2 with a unique ordered pair.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$.

The notation used is $(x, y) \to (x + 3, y)$. For example, $(3, 2) \to (3 + 3, 2)$, i.e. $(6, 2)$. This second ordered pair is uniquely determined by the first.

In the diagram it is seen that the point with coordinates $(3, 2)$ is mapped to the point with coordinates $(6, 2)$.

Also $(-2, 4) \to (1, 4)$ and $(-1, -2) \to (2, -2)$.



In applying a transformation it is useful to think of every point (x, y) on the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written with the translation considered above:

$$x' = x + 3 \text{ and } y' = y$$

The translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis can be described by the rule $(x, y) \to (x + 2, y + 4)$.

[1] These sections could be omitted but they form a sound foundation for further study.

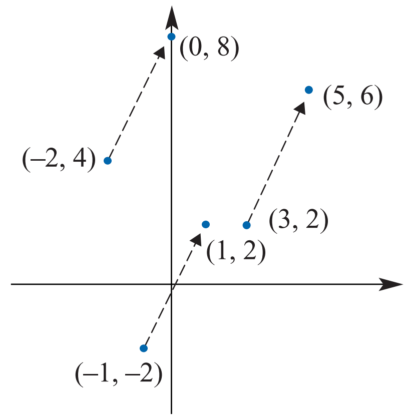
For example, $(3, 2) \to (3 + 2, 2 + 4)$.

The following can be written for this second translation considered above:

$$x' = x + 2 \text{ and } y' = y + 4$$

In the diagram it is seen that the point with coordinates (3, 2) is mapped to the point with coordinates (5, 6).

Also $(-2, 4) \to (0, 8)$ and $(-1, -2) \to (1, 2)$.



In general

- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \to (x + h,\ y + k)$$

or $\quad x' = x + h \quad$ and $\quad y' = y + k$

where h and k are positive numbers.

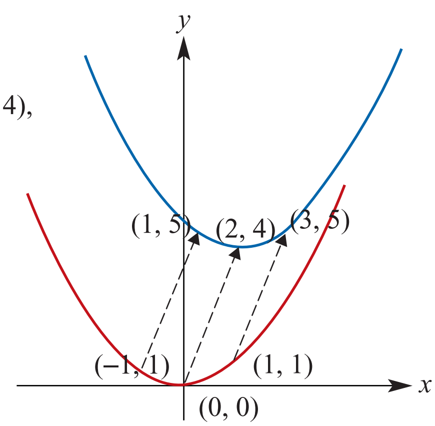
- A translation in the negative direction of the x-axis and the negative direction of the y-axis is described by

$$(x, y) \to (x - h,\ y - k)$$

or $\quad x' = x - h \quad$ and $\quad y' = y - k$

where h and k are positive numbers.

Consider translating the set of points defined by a function such as $\{(x, y): y = x^2\}$. Now with the transformation defined by the rule $(x, y) \to (x + 2, y + 4)$, it is known that $x' = x + 2$ and $y' = y + 4$ and hence $x = x' - 2$ and $y = y' - 4$. Thus $\{(x, y): y = x^2\}$ maps to $\{(x', y'): y' - 4 = (x' - 2)^2\}$. This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.



In general, the curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$ by the translation with rule $(x, y) \to (x + h, x + k)$.

This generalisation can now be applied to other functions.

Example 19

Find the image of the curve with equation $y = f(x)$, where $f(x) = \dfrac{1}{x}$ under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

The rule is $(x, y) \to (x + 3, y - 2)$. Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$. Then $x' = x + 3$ and $y' = y - 2$ and hence $x = x' - 3$ and $y = y' + 2$. The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$, i.e. $y = \dfrac{1}{x}$ is mapped to $y' + 2 = \dfrac{1}{x' - 3}$.

Using a graphics calculator

Example 20

Use a graphics calculator to sketch the graph of the function obtained from the graph of the function with equation $y = \sqrt{x}$ by a translation of:

a 2 units in the positive direction of the x-axis

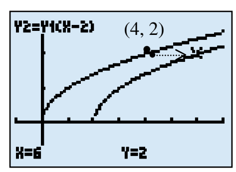
b 3 units in the negative direction of the y-axis

Solution

a A translation of h units ($h > 0$) in the positive direction of the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(x - h)$.

Therefore $\quad \text{Y1} = \sqrt{\text{X}} \qquad (= f(x))$

and $\quad \text{Y2} = \text{Y1}(\text{X} - 2) \qquad (= f(x - h))$

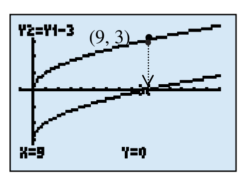


The transformed graph of Y2 is shown to the right, together with the graph of the original function.

b A translation of k units ($k > 0$) in the negative direction of the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(x) - k$.

Therefore $\quad \text{Y1} = \sqrt{\text{X}} \qquad (= f(x))$

and $\quad \text{Y2} = \text{Y1} - 3 \qquad (= f(x) - k)$



The transformed graph of Y2 is shown to the right, together with the graph of the original function.

Applying translations to sketch graphs

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of $y = \dfrac{1}{x-2}$ note that it is of the form $y = f(x-2)$ where $f(x) = \dfrac{1}{x}$. That is, the graph of $y = \dfrac{1}{x}$ is translated 2 units in the positive direction of the x-axis. Examples of the graphs of two other functions to which this translation is applied are:

- $f(x) = x^2$ $\quad f(x-2) = (x-2)^2$
- $f(x) = \sqrt{x}$ $\quad f(x-2) = \sqrt{x-2}$

Exercise 5H

1 Sketch the graphs of each of the following, labelling asymptotes and axes intercepts:

a $y = \dfrac{1}{x} + 3$ **b** $y = \dfrac{1}{x^2} - 3$ **c** $y = \dfrac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \dfrac{1}{x-1}$ **f** $y = \dfrac{1}{x} - 4$

g $y = \dfrac{1}{x+2}$ **h** $y = \dfrac{1}{x-3}$ **i** $f(x) = \dfrac{1}{(x-3)^2}$

j $f(x) = \dfrac{1}{(x+4)^2}$ **k** $f(x) = \dfrac{1}{x-1} + 1$ **l** $f(x) = \dfrac{1}{x-2} + 2$

Example 19

2 For $y = f(x) = \dfrac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axes intercepts:

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

3 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axes intercepts:

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

4 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axes intercepts:

a $y = f(x-1) + 2$ **b** $y = f(x-3) + 1$ **c** $y = f(x+3) - 5$

d $y = f(x+1) - 3$ **e** $y + 2 = f(x+1)$ **f** $y = f(x-5) - 1$

5 Sketch the graphs of each of the following, stating the equations of asymptotes, the axes intercepts and the range of each function:

a $y = \dfrac{1}{x^2} + 1$ **b** $y = \dfrac{3}{x^2}$ **c** $y = \dfrac{1}{(x-1)^2}$ **d** $y = \dfrac{1}{x^2} - 4$

5.9 Dilations and reflections

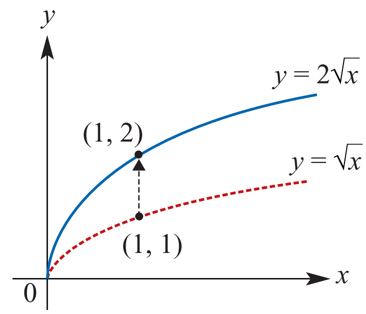
Dilations

A dilation of factor 2 from the x-axis can be defined by the rule $(x, y) \rightarrow (x, 2y)$. Hence the point with coordinates $(1, 1) \rightarrow (1, 2)$.

A dilation of factor 2 from the y-axis can be defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Dilation from the x-axis

The curve with equation $y = \sqrt{x}$ is considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under dilation of factor 2 from the x-axis. Hence $x' = x$ and $y' = 2y$. Then $x = x'$ and $y = \frac{y'}{2}$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\frac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.



In general

- A dilation of a units from the x-axis is described by the rule

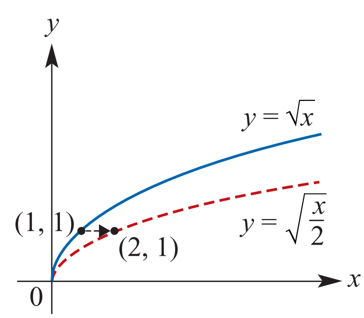
  $$(x, y) \to (x, ay)$$

  or $\quad x' = x \quad$ and $\quad y' = ay$

  where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = af(x)$ by the transformation with rule $(x, y) \to (x, ay)$.

Dilation from the y-axis

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under dilation of factor 2 from the y-axis. Hence $x' = 2x$ and $y' = y$. Then $x = \frac{x'}{2}$ and $y = y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



In general

- A dilation of a units from the y-axis is described by the rule

  $$(x, y) \to (ax, y)$$

  or $\quad x' = ax \quad$ and $\quad y' = y$

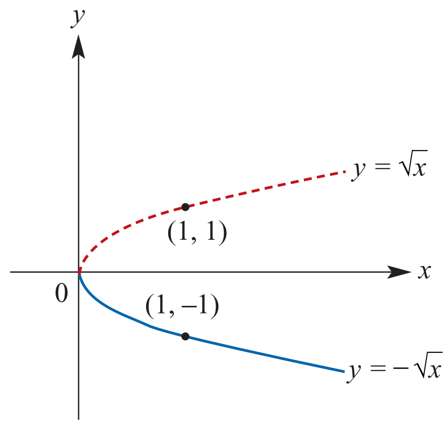
  where a is a positive number.
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$ by the transformation with rule $(x, y) \to (ax, y)$.

Reflections in the axes

Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under a reflection in the x-axis. Hence $x' = x$ and $y' = -y$. Then $x = x'$ and $y = -y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x}$, i.e. the curve with equation $y = -\sqrt{x}$.



In general

- A reflection in the x-axis is described by the rule

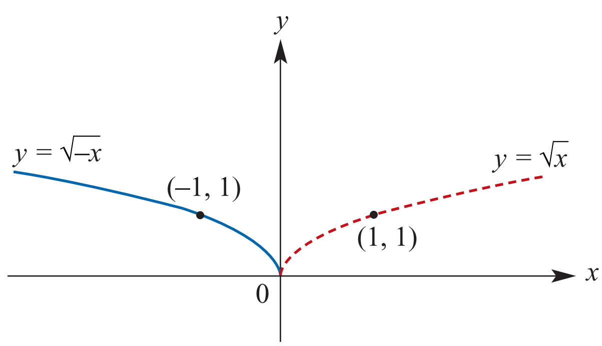
  $$(x, y) \to (x, -y)$$

  or $\quad x' = x \quad$ and $\quad y' = -y$
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$ by the transformation with rule $(x, y) \to (x, -y)$.

Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

The curve with equation $y = \sqrt{x}$ is again considered. Let (x', y') be the image of the point with coordinates (x, y) on the curve with equation $y = \sqrt{x}$ under a reflection in the y-axis. Hence $x' = -x$ and $y' = y$. Then $x = -x'$ and $y = y'$ and the curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



In general

- A reflection in the y-axis is described by the rule

  $$(x, y) \to (-x, y)$$

  or $\quad x' = -x \quad$ and $\quad y' = y$
- The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$ by the transformation with rule $(x, y) \to (-x, y)$.

Example 21

Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 4:

a from the y-axis **b** from the x-axis

Solution

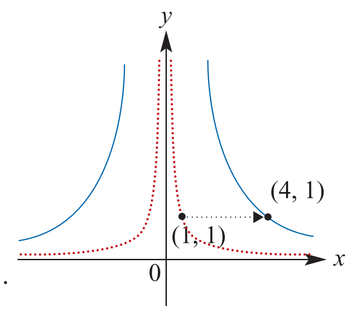
a $(x, y) \rightarrow (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x$, $y' = y$.

Rearranging gives $x = \frac{x'}{4}$, $y = y'$.

Therefore $y = \frac{1}{x^2}$ becomes $y' = \frac{1}{\left(\frac{x'}{4}\right)^2}$.

So the rule of the transformed function is $y = \frac{16}{x^2}$.



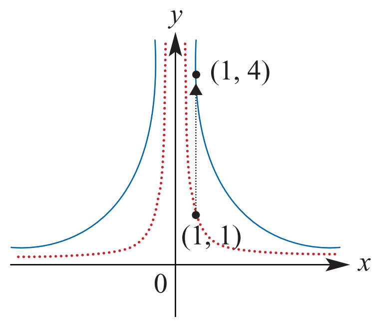
b $(x, y) \rightarrow (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 4y$.

Rearranging gives $x = x'$, $y = \frac{y'}{4}$.

Therefore $y = \frac{1}{x^2}$ becomes $\frac{y'}{4} = \frac{1}{(x')^2}$.

So the rule of the transformed function is $y = \frac{4}{x^2}$.



Using a graphics calculator

Example 22

Use the graphics calculator to sketch the graph of the function obtained when the graph of the function with equation $y = \sqrt{x}$ is dilated by factor:

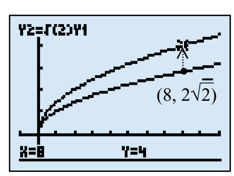
a $\sqrt{2}$ from the x-axis **b** $\frac{1}{2}$ from the y-axis

Solution

a A dilation of factor k from the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = kf(x)$.

Therefore $\quad \text{Y1} = \sqrt{\text{X}} \quad (= f(x))$

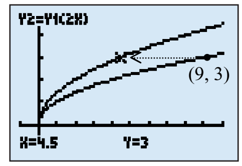
and $\quad \text{Y2} = \sqrt{(2)}\text{Y1} \quad (= kf(x))$



The transformed graph of Y2 is shown to the right, together with the original function.

Note: Y1 is obtained by pressing $\boxed{\text{VARS}}$ and selecting the **Y-VARS** submenu, then **1:Function** and finally Y1.

b A dilation of factor k from the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(\frac{x}{k})$.



Therefore $\quad \text{Y1} = \sqrt{\text{X}} \qquad (= f(x))$

and $\quad \text{Y2} = \text{Y1(2X)} \quad \left(= f\left(\frac{x}{k}\right)\right)$

The transformed graph of Y2 is shown to the right, together with the original function.

Note: In this example the dilation of factor $\sqrt{2}$ from the x-axis has produced the same graph as a dilation of factor $\frac{1}{2}$ from the y-axis.

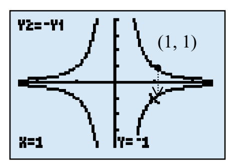
Example 23

Use the graphics calculator to sketch the graph of the function obtained when the graph of the function with equation $y = \dfrac{1}{x^2}$ is reflected in:

a the x-axis **b** the y-axis

Solution

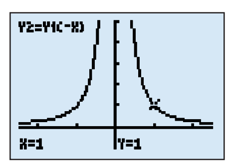
a A reflection in the x-axis transforms the function with rule $y = f(x)$ into the function with rule $y = -f(x)$.



Therefore $\quad \text{Y1} = 1/\text{X}^2 \quad (= f(x))$

and $\quad \text{Y2} = -\text{Y1} \quad (= -f(x))$

b A reflection in the y-axis transforms the function with rule $y = f(x)$ into the function with rule $y = f(-x)$.



Therefore $\quad \text{Y1} = 1/\text{X}^2 \qquad (= f(x))$

and $\quad \text{Y2} = \text{Y1}(-\text{X}) \quad (= f(-x))$

The transformed graph of Y2 is shown to the right, together with the original function.

Note: The reflection in the y-axis has produced the graph of the original function.

Applying dilations and reflections to sketch graphs

In order to sketch the graph of $y = \sqrt{\frac{x}{2}}$ note that it is of the form $y = f\left(\frac{x}{2}\right)$, where $f(x) = \sqrt{x}$. This is the graph of $y = \sqrt{x}$ dilated by a factor 2 from the y-axis. Examples of other functions under this dilation are:

- $f(x) = x^2, \quad f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$
- $f(x) = \frac{1}{x}, \quad f\left(\frac{x}{2}\right) = \frac{1}{x/2} = \frac{2}{x}$

It should be noted that each of these functions formed by a dilation of factor 2 from the y-axis can also be formed by a dilation from the x-axis. This result is not true in general, as will be seen when new functions are introduced in chapters 12 and 13.

- $y = \sqrt{\frac{x}{2}} = \frac{1}{\sqrt{2}}\sqrt{x} = \frac{1}{\sqrt{2}} f(x)$, where $f(x) = \sqrt{x}$. That is, it is formed by a dilation of factor $\frac{1}{\sqrt{2}}$ from the x-axis.
- $y = \frac{x^2}{4} = \frac{1}{4}x^2 = \frac{1}{4} f(x)$, where $f(x) = x^2$. That is, it is formed by a dilation of factor $\frac{1}{4}$ from the x-axis.
- $y = \frac{2}{x} = 2\frac{1}{x} = 2f(x)$, where $f(x) = \frac{1}{x}$. That is, it is formed by a dilation of factor 2 from the x-axis.

Exercise 5I

Example 21

1 Write down the equation of the rule obtained when the graph of each of the functions below is transformed by:

i a dilation of factor $\frac{1}{2}$ from the y-axis
ii a dilation of factor 5 from the y-axis
iii a dilation of factor $\frac{2}{3}$ from the x-axis
iv a dilation of factor 4 from the x-axis
v a reflection in the x-axis
vi a reflection in the y-axis

a $y = x^2$
b $y = \frac{1}{x^2}$
c $y = \frac{1}{x}$
d $y = \sqrt{x}$

2 Sketch the graphs of each of the following:

a $y = 3\sqrt{x}$
b $y = -\frac{1}{x}$
c $y = \frac{3}{x}$
d $y = \frac{1}{2x}$
e $y = \sqrt{3x}$
f $y = \frac{3}{2x}$

5.10 Combinations of transformations

In this section sequences of transformations are applied.

For example, first consider:

- a dilation of factor 2 from the x-axis followed by
- a reflection in the x-axis.

The rule becomes $(x, y) \to (x, 2y) \to (x, -2y)$. First the dilation and then the reflection is applied.

For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis followed by
- a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$. First the dilation and then the translation is applied.

For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 24

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

Solution

a From the discussion above, $(x, y) \to (x, 2y) \to (x, -2y)$. Hence if (x', y') is the image of (x, y) under this map, $x' = x$ and $y' = -2y$. Hence $x = x'$ and $y = \dfrac{y'}{-2}$. The graph of the image will have equation $\dfrac{y'}{-2} = \sqrt{x'}$ and hence $y' = -2\sqrt{x'}$.

b From the discussion above, $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$. Hence if (x', y') is the image of (x, y) under this map, $x' = x + 2$ and $y' = 2y - 3$. Hence $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$. Thus the graph of the image will have equation $\dfrac{y' + 3}{2} = \sqrt{x' - 2}$ or $y' = 2\sqrt{x' - 2} - 3$.

Determining transformations

The method that has been used to find the effect of transformations can be reversed to determine the sequence of transformations used to take a graph to its image. For the example above, in order to find the sequence of transformations which map $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution.

$$y = \sqrt{x} \text{ maps to } \frac{y'}{-2} = \sqrt{x'}$$

Hence $x = x'$ and $y = \dfrac{y'}{-2}$.

Therefore $x' = x$ and $y' = -2y$.

The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, as you recognise the form of the image. For the combination of transformations in this course it is often simpler to do this.

Example 25

Find a sequence of transformations which take the graph of $y = x^2$ to the graph of $y = 2(x - 2)^2 + 3$.

Solution

By inspection

By inspection it is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

By the method

$$y = x^2 \text{ maps to } y' = 2(x' - 2)^2 + 3$$

Rearranging the expression on the right

$$\frac{y' - 3}{2} = (x' - 2)^2$$

It can be seen that $y = \dfrac{y' - 3}{2}$ and $x = x' - 2$. Solving for x' and y' gives $y' = 2y + 3$ and $x' = x + 2$.

The transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

Exercise 5J

Example 24

1 Find the equation of the image of the graph $y = \sqrt{x}$ when each of the following sequences of transformations have been applied:

- **a** a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 from the x-axis
- **b** a translation of 3 units in the negative direction of the x-axis followed by a reflection in the x-axis
- **c** a reflection in the x-axis followed by a dilation of factor 3 from the x-axis
- **d** a reflection in the x-axis followed by a dilation of factor 2 from the y-axis

e a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

f a dilation of factor 2 from the y-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

2 Repeat Question 1 for $y = \frac{1}{x}$.

Example 25 **3** For each of the following find a sequence of transformations that take:

a the graph of $y = x^2$ to the graph of

i $y = 2(x-1)^2 + 3$ **ii** $y = -(x+1)^2 + 2$ **iii** $y = (2x+1)^2 - 2$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{2}{x+3}$ **ii** $y = \frac{1}{x+3} + 2$ **iii** $y = \frac{1}{x-3} - 2$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x+3} + 2$ **ii** $y = 2\sqrt{3x}$ **iii** $y = -\sqrt{x} + 2$

5.11 Functions and modelling exercises

Example 26

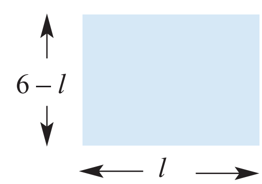
A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of the length, l. By sketching a graph find the maximum area that can be fenced.

Solution

If l = length of the enclosure

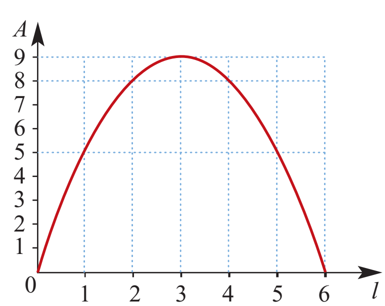
$$\text{then width} = \frac{12-2l}{2} = 6 - l$$

$$\therefore \text{ area } A(l) = l(6-l) = 6l - l^2$$



The domain of A is the interval (0, 6).

The maximum area is 9 m^2 and occurs when $l = 3$ m, i.e. the enclosure is a square.



Example 27

The following list shows Australia Post airmail rates for articles:

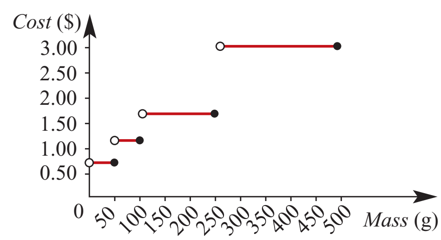
Mass (m, g)	Cost (C, \$)
Up to 50 g	\$0.70
Over 50 g up to 100 g	\$1.15
Over 100 g up to 250 g	\$1.70
Over 250 g up to 500 g	\$3.00

Sketch a graph of the cost function, C, giving its domain and range and the rules that define it.

Solution

The rules are
$$\begin{aligned} C &= 0.70 \text{ for } 0 < m \le 50 \\ &= 1.15 \text{ for } 50 < m \le 100 \\ &= 1.70 \text{ for } 100 < m \le 250 \\ &= 3.00 \text{ for } 250 < m \le 500 \end{aligned}$$

The graph is as follows:



Range $= \{0.70, 1.15, 1.70, 3.00\}$

Example 28

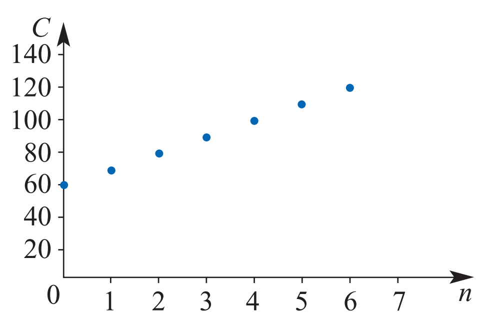
A book club has a membership fee of \$60.00 and each book purchased is \$10.00. Construct a cost function that can be used to determine the cost of different numbers of books, then sketch its graph.

Solution

Let C denote the cost (in dollars) and n denote the number of books purchased, then

$$C = 60 + 10n$$

The domain of this function is $Z^+ \cup \{0\}$, the set of positive integers, and its graph will be as shown.



The range of this function is $\{x: x \in Z \text{ and } x \geq 60\}$. Sometimes to simplify the situation we represent such functions by a continuous line. Strictly, this is not mathematically correct but may aid our understanding of the situation.

Exercise 5K

Example 26

1 **a** **i** Find an expression for the area A in terms of x and y.

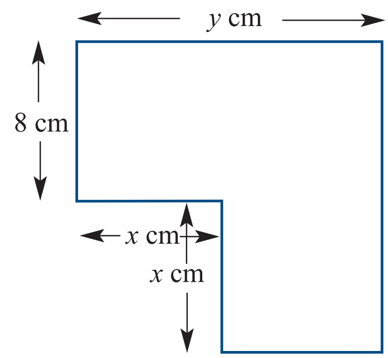
ii Find an expression for the perimeter P in terms of x and y.

b **i** If $P = 64$ cm, find A in terms of x.

ii Find the allowable values for x.

iii Sketch the graph of A against x for these values.

iv What is the maximum area?



Example 27

2 Suppose Australia Post charged the following rates for airmail letters to Africa: \$1.20 up to 20 g; \$2.00 over 20 g and up to 50 g; \$3.00 over 50 g and up to 150 g.

a Write a cost function, C (\$), in terms of the mass, m (g) for letters up to 150 g.

b Sketch the graph of the function, stating the domain and range.

3 Telenet listed the following scale of charges for a 3 minute STD call between the hours of 6 pm and 10 pm on Monday to Friday.

Distance, d (km)	Up to 25 km (not including 25)	25 up to 50 km (not including 50)	50 up to 85 km (not including 85)	85 up to 165 km (not including 165)	165 up to 745 km (not including 745)	745 km and over
Cost, C (\$)	0.30	0.40	0.70	1.05	1.22	1.77

a Write a cost function, C (\$), in terms of distance, d (km).

b Sketch the graph of the function.

4 Self-Travel, a car rental firm, has two methods of charging for car rental:

i Method 1: \$64 per day + 25 cents per kilometre

ii Method 2: \$89 per day with unlimited travel

a Write a rule for each method if C_1 is the cost, in \$, using method 1 for x kilometres travelled, and C_2 is the cost using method 2.

b Draw a graph of each rule the same axes.

c Determine, from the graph, the distance which must be travelled per day if method 2 is cheaper than method 1.

Review

Chapter summary

- **Set notation**

  $x \in A$ x is an element of A

  $x \notin A$ x is not an element of A

  $A \subseteq B$ A is a subset of B

  $A \cap B$ $x \in A \cap B$ if and only if $x \in A$ and $x \in B$

  $A \cup B$ $x \in A \cup B$ if and only if $x \in A$ or $x \in B$

  $A \setminus B$ $\{x: x \in A, x \notin B\}$
- **Sets of numbers**

  N Natural numbers

  Z Integers

  Q Rational numbers

  R Real numbers
- **Interval notation**

  $(a, b) = \{x: a < x < b\}$ $[a, b] = \{x: a \leq x \leq b\}$

  $(a, b] = \{x: a < x \leq b\}$ $[a, b) = \{x: a \leq x < b\}$

  $(a, \infty) = \{x: a < x\}$ $[a, \infty) = \{x: a \leq x\}$

  $(-\infty, b) = \{x: x < b\}$ $(-\infty, b] = \{x: x \leq b\}$
- A **relation** is a set of ordered pairs.

  The **domain** of a relation is the set of all first elements of the ordered pairs of the relation.

  The **range** of a relation is the set of all second elements of the ordered pairs of the relation.
- For a function f and an element x of the domain of f there is unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f.

  If $(x, y) \in f$, then x is called a **pre-image** of y.
- **One-to-one function**

  A function is said to be one-to-one if, for $a, b \in \text{dom } f$, $a \neq b$, then $f(a) \neq f(b)$.

  In other words f is called one-to-one if every image under f has a unique pre-image.
- **Implied domain**

  The implied domain of a function is the largest subset of R for which the rule is defined.
- **Restrictions of a function**

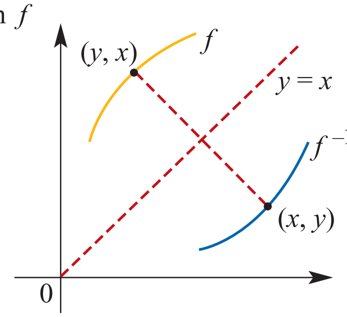
  For a function f with domain D a new function g may be defined with domain $A \subseteq D$ and rule defined by $g(x) = f(x)$ for all $x \in A$. The function g is called a **restriction** of f.

- **Inverses of functions**

  If f is a one-to-one function then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.
  Thus if f is a one-to-one function, a new function f^{-1}, called the inverse of f, may be defined by:

  $$f^{-1}(x) = y, \text{ if } f(y) = x, \text{ for } x \in \text{ran } f, y \in \text{dom } f$$

  The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f.
  Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.

  From this the following is evident:

  $$\text{dom } f^{-1} = \text{ran } f$$
  $$\text{ran } f^{-1} = \text{dom } f$$

- **Transformations of the graphs of functions**

  In general, a **translation** of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$ or $x' = x + h$ and $y' = y + k$, where h and k are positive numbers.
  For **translations** in the negative direction of the x-axis, $(x, y) \to (x - h, y - k)$ or $x' = x - h$ and $y' = y - k$, where h and k are positive numbers.
  In general:
  - The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$ by the **translation** with rule $(x, y) \to (x + h, y + k)$.
  - A **dilation of *a* units from the *x*-axis** is described by the rule $(x, y) \to (x, ay)$ or $x' = x$ and $y' = ay$, where a is a positive number.
  - The curve with equation $y = f(x)$ is mapped to the curve with equation $y = af(x)$ by the **dilation** with rule $(x, y) \to (x, ay)$.
  - A **dilation of *a* units from the *y*-axis** is described by the rule $(x, y) \to (ax, y)$ or $x' = ax$ and $y' = y$, where a is a positive number.
  - The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$ by the **dilation** with rule $(x, y) \to (ax, y)$.
  - A **reflection in the *x*-axis** is described by the rule $(x, y) \to (x, -y)$ or $x' = x$ and $y' = -y$.
  - The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$ by the **reflection** with rule $(x, y) \to (x, -y)$.
  - A **reflection in the *y*-axis** is described by the rule $(x, y) \to (-x, y)$ or $x' = -x$ and $y' = y$.
  - The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$ by the **reflection** with rule $(x, y) \to (-x, y)$.

Multiple-choice questions

1 For $f(x) = 10x^2 + 2$, $f(2a)$ equals

A $20a^2 + 2$ **B** $40a^2 + 2$ **C** $2a^2 + 2a$ **D** $100a^2 + 2$ **E** $10a^2 + 2a$

2 The maximal domain of the function f with rule $f(x) = \sqrt{3x + 5}$ is

A $(0, \infty)$ **B** $\left(-\frac{5}{3}, \infty\right)$ **C** $(5, \infty)$ **D** $[-5, \infty)$ **E** $\left[-\frac{5}{3}, \infty\right)$

3 The range of the relation $x^2 + y^2 > 9$ when $x, y \in R$ is

A $[0, \infty)$ **B** R **C** $(-\infty, 0]$ **D** $(3, \infty) \cup (-\infty, -3)$ **E** $(-\infty, 0)$

4 For $f(x) = 7x - 6$, $f^{-1}(x)$ equals

A $7x + 4$ **B** $\frac{1}{7}x + 6$ **C** $\frac{1}{7}x + \frac{6}{7}$ **D** $\frac{1}{7x - 6}$ **E** $\frac{1}{7}x - 6$

5 For $f: (a, b] \to R$, $f(x) = 3 - x$

A $(3 - a, 3 - b)$ **B** $(3 - a, 3 - b]$ **C** $(3 - b, 3 - a)$

D $(3 - b, 3 - a]$ **E** $[3 - b, 3 - a)$

6 Which of the following functions is not one-to-one?

A $f(x) = 9 - x^2, x \geq 0$ **B** $f(x) = \sqrt{9 - x^2}$ **C** $f(x) = 1 - 9x$

D $f(x) = \sqrt{x}$ **E** $f(x) = \frac{9}{x}$

7 The graph of $y = \frac{2}{x} + 3$ is reflected in the x-axis and then in the y-axis. The equation of the final image is

A $y = -\frac{2}{x} + 3$ **B** $y = -\frac{2}{x} - 3$ **C** $y = \frac{2}{x} + 3$

D $y = \frac{2}{x} - 3$ **E** $y = 2x - 3$

8 The sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = -(2x - 6)^2 + 4$ is

A a reflection in the y-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis

B a reflection in the y-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

C a reflection in the y-axis followed by a dilation of 2 from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

D a reflection in the x-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the negative direction of the x-axis and 4 units in the positive direction of the y-axis

E a reflection in the x-axis followed by a dilation of $\frac{1}{2}$ from the y-axis and then a translation of 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis

9 For $f: [-1, 5) \to R,\ f(x) = x^2$, the range is

A R **B** $[0, \infty)$ **C** $[0, 25)$ **D** $[1, 25]$ **E** $[0, 5]$

10 Which of the following rules does **not** describe a function?

A $y = x^2 - x$ **B** $y = \sqrt{4 - x^2}$ **C** $y = 3, x > 0$ **D** $x = 3$ **E** $y = 3x$

Short-answer questions (technology-free)

1 If f is the function with rule $f(x) = 2 - 6x$, find:

a $f(3)$ **b** $f(-4)$ **c** the value of x for which f maps x to 6

2 For $[-1, 6] \to R,\ f(x) = 6 - x$:

a sketch the graph of f **b** state the range of f

3 Sketch the graphs of each of the following, stating the range of each:

a $\{(x, y): 3x + y = 6\}$ **b** $\{(x, y): y = 3x - 2, x \in [-1, 2]\}$

c $\{(x, y): y = x^2, x \in [-2, 2]\}$ **d** $\{(x, y): y = 9 - x^2\}$

e $\{(x, y): y = x^2 + 4x + 6\}$ **f** $\{(1, 2)\ (3, 4)\ (2, -6)\}$

g $f: R \to R,\ f(x) = (x - 2)^2$ **h** $f: R \setminus \{0\} \to R,\ f(x) = \dfrac{1}{x} + 2$

i $\left(x - \dfrac{1}{2}\right)^2 + (y + 2)^2 = 9$ **j** $f: [-1, 3] \to R,\ f(x) = x$

4 The function f has rule $f(x) = \dfrac{a}{x} + b$ such that $f(1) = \dfrac{3}{2}$ and $f(2) = 9$.

a Find the values of a and b. **b** State the implied domain of f.

5 Given that $f: [0, 2] \to R,\ f(x) = 2x - x^2$:

a sketch the graph **b** state the range

6 Given that $f(x) = ax + b,\ f(5) = 10$ and $f(1) = -2$, find the values of a and b.

7 Given that $f(x) = ax^2 + bx + c,\ f(0) = 0,\ f(4) = 0$ and $f(-2) = -6$, find the values of a, b and c.

8 State the implied (maximal) domain for each of the following:

a $y = \dfrac{1}{x - 2}$ **b** $f(x) = \sqrt{x - 2}$ **c** $y = \sqrt{25 - x^2}$

d $f(x) = \dfrac{1}{2x - 1}$ **e** $g(x) = \sqrt{100 - x^2}$ **f** $h(x) = \sqrt{4 - x}$

9 State which of the following functions are one-to-one:

a $y = x^2 + 2x + 3$ **b** $f: [2, \infty) \to R, f(x) = (x - 2)^2$

c $f(x) = 3x + 2$ **d** $f(x) = \sqrt{x - 2}$

e $f(x) = \dfrac{1}{x - 2}$ **f** $f: [-1, \infty) \to R, f(x) = (x + 2)^2$

g $f: [-3, 5] \to R, f(x) = 3x - 2$ **h** $f(x) = 7 - x^2$

i $f(x) = \dfrac{1}{(x - 2)^2}$ **j** $h(x) = \dfrac{1}{x - 2} + 4$

10 Sketch the graphs of each of the following:

a $f(x) = \begin{cases} 3x - 1, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ 9, & x \in (-\infty, -3) \end{cases}$ **b** $h(x) = \begin{cases} 1 - 2x, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ -x^2, & x \in (-\infty, -3) \end{cases}$

11 For each of the following find the inverse function, stating the rule and the domain:

a $f: [-1, 5] \to R, f(x) = 3x - 2$ **b** $f: [-2, \infty) \to R, f(x) = \sqrt{x+2} + 2$

c $f: [-1, \infty) \to R, f(x) = 3(x+1)^2$ **d** $f: (-\infty, 1) \to R, f(x) = (x-1)^2$

12 For the function f with rule $f(x) = \sqrt{x}$, find the equation for the graph of the image under each of the following transformations:

a a translation of 2 in the positive direction of the x-axis and 3 in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis **c** a reflection in the x-axis

d a reflection in the y-axis **e** a dilation of factor 3 from the y-axis

Extended-response questions

1 An Easyride coach leaves town X and maintains a constant speed of 80 km/h for 4 hours, stops at town Y for $\frac{3}{4}$ hour before travelling for a further $2\frac{1}{2}$ hours at 80 km/h to its destination at town Z. A second coach leaves town Z at the same time and runs express to town X, completing its journey in $5\frac{1}{2}$ hours.

a Construct functions that describe the distance, d km, from X of each coach at time t, stating the domain, range and rule for each.

b Calculate the distance, from X, at which the two coaches pass each other.

2 A parking meter is designed to accept 200 twenty-cent coins.

a Write a rule which gives the number of hours parking, P hours, in terms of n, the number of twenty-cent coins inserted, when the cost of parking is 20c for each half hour.

b Sketch the function, stating the domain and range.

3 The following table shows the amount of taxation payable for a given range of annual income from 1 July 1986.

Taxable income, $	Tax	
\$1 to \$ $(C - 1)$	Nil	
C to 12 499	Nil	plus 24.42 cents for each \$1 over C
\$12 500 to 12 599	D	plus 26.50 cents for each \$1 over 12 500
\$12 600 to 19 499	D + \$26.50	plus 29.42 cents for each \$1 over 12 600
\$19 500 to 27 999	D + \$2056.48	plus 44.25 cents for each \$1 over \$ 19 500
\$28 000 to 34 999	D + \$5817.73	plus 46.83 cents for each \$1 over \$28 000
\$35 000 and over	D + \$9095.83	plus 57.08 cents for each \$1 over \$35 000

C is an amount called the reduced taxfree threshold
and $D = 24.42\%$ of $(12\,500 - C)$
$= 0.2442\,(12\,500 - C)$

a If for a particular taxpayer C is \$4223, write a rule for the tax payable, T (\$), for a taxable income, x (\$), where $29\,000 \le x \le 33\,000$.

b Sketch a graph of this function, stating the range.

c Calculate the amount of tax payable on an income of \$30 000.

4 The Exhibition Centre hires a graphics company to produce a poster for an exhibit. The graphics company charges \$1000 and an additional \$5 for each poster produced.

a **i** Write the rule for a function, $C(n)$, which describes the cost to the Exhibition Centre of obtaining n posters.

ii Sketch the graph of C against n (use a continuous model).

b The Exhibition Centre is going to sell the posters for \$15.00 each.

i Write down the rule for the function $P(n)$ which gives the profit when the Exhibition Centre sells n posters.

ii Sketch the graph of this function (use a continuous model).

5 An article depreciates by 5% of its original cost each year. If the original cost was \$8000, find an expression for the value, V, of the item n years after purchase.

6 The organisers of a sporting event know that, on average, 50 000 people will visit the venue each day. They are presently charging \$15.00 for an admission ticket. Each time in the past when they have raised the admission price an average of 2500 fewer people have come to the venue for each \$1.00 increase in ticket price. Let x represent the number of \$1.00 increases.

a Write the rule for a function which gives the revenue, R, in terms of x.

b Sketch the graph of R against x.

c Find the price which will maximise the revenue.

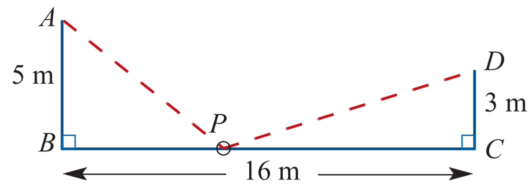
7 A thin wire of length a cm is bent to form the perimeter of a pentagon $ABCDE$ in which $BCDE$ is a rectangle, and ABE is an equilateral triangle. Let x cm be the length of CD and $A(x)$ be the area of the pentagon.

a Find $A(x)$ in terms of x.

b State the allowable values for x.

c Show that the maximum area $= \dfrac{a^2}{4(6-\sqrt{3})}$ cm^2.

8 Let P be a point between B and C on line BC.



Let $d(x)$ be the distance $(PA + PD)$ m, where x is the distance of P from B.

a **i** Find an expression for $d(x)$.

ii Find the allowable values of x.

b **i** Use a graphics calculator to plot the graph of $y = d(x)$ for a suitable window setting.

ii Find the value of x if $d(x) = 20$ (correct to 2 decimal places).

iii Find the values of x for which $d(x) = 19$ (correct to 2 decimal places).

c **i** Find the minimum value of $d(x)$ and the value of x for which this occurs.

ii State the range of the function.

9 **a** Find the coordinates of $A(x_1, y_1)$, and $B\ (x_1, y_1)$.

b Let $d(x)$ be the 'vertical' distance between the graphs for $x \in [x_2, x_1]$.

i Find $d(x)$ in terms of x.

ii Plot the graph of $d(x)$ against x for $x \in [x_1, x_2]$ and on the same screen the graphs of $y = 2x$ and $y = (x + 1)(6 - x)$.

c **i** State the maximum value of the function defined by $d(x)$ for $x \in [x_2, x_1]$.

ii State the range of this function.

d Repeat with the graphs $y = 5x$ and $y = (x + 1)(6 - x)$.